

# CYCLICALLY FIVE-CONNECTED CUBIC GRAPHS

Neil Robertson<sup>\*1</sup>  
Department of Mathematics  
Ohio State University  
231 W. 18th Ave.  
Columbus, Ohio 43210, USA

P. D. Seymour<sup>2</sup>  
Department of Mathematics  
Princeton University  
Princeton, New Jersey 08544, USA

and

Robin Thomas<sup>\*3</sup>  
School of Mathematics  
Georgia Institute of Technology  
Atlanta, Georgia 30332, USA

23 February 1995  
Revised 4 March 2015

---

<sup>\*</sup> Research partially performed under a consulting agreement with Bellcore, and partially supported by DIMACS Center, Rutgers University, New Brunswick, New Jersey 08903, USA.

<sup>1</sup> Partially supported by NSF under Grant No. DMS-8903132 and by ONR under Grant No. N00014-92-J-1965.

<sup>2</sup> This research was performed while the author was employed at Bellcore, 445 South St., Morristown, NJ 07960.

<sup>3</sup> Partially supported by NSF under Grants No. DMS-9303761 and DMS-1202640 and by ONR under Grant No. N00014-93-1-0325.

## ABSTRACT

A cubic graph  $G$  is *cyclically 5-connected* if  $G$  is simple, 3-connected, has at least 10 vertices and for every set  $F$  of edges of size at most four, at most one component of  $G \setminus F$  contains circuits. We prove that if  $G$  and  $H$  are cyclically 5-connected cubic graphs and  $H$  topologically contains  $G$ , then either  $G$  and  $H$  are isomorphic, or (modulo well-described exceptions) there exists a cyclically 5-connected cubic graph  $G'$  such that  $H$  topologically contains  $G'$  and  $G'$  is obtained from  $G$  in one of the following two ways. Either  $G'$  is obtained from  $G$  by subdividing two distinct edges of  $G$  and joining the two new vertices by an edge, or  $G'$  is obtained from  $G$  by subdividing each edge of a circuit of length five and joining the new vertices by a matching to a new circuit of length five disjoint from  $G$  in such a way that the cyclic orders of the two circuits agree. We prove a companion result, where by slightly increasing the connectivity of  $H$  we are able to eliminate the second construction. We also prove versions of both of these results when  $G$  is almost cyclically 5-connected in the sense that it satisfies the definition except for 4-edge cuts such that one side is a circuit of length four. In this case  $G'$  is required to be almost cyclically 5-connected and to have fewer circuits of length four than  $G$ . In particular, if  $G$  has at most one circuit of length four, then  $G'$  is required to be cyclically 5-connected. However, in this more general setting the operations describing the possible graphs  $G'$  are more complicated.

## 1. INTRODUCTION

The primary motivation for this work comes from Tutte’s 3–edge-coloring conjecture [12], the following (definitions are given later).

**(1.1) Conjecture.** *Every 2–edge-connected cubic graph that does not topologically contain the Petersen graph is 3–edge-colorable.*

Our strategy is to reduce (1.1) to “apex” and “doublecross” graphs, two classes of graphs that are close to planar graphs, and then modify our proof of the Four Color Theorem [7] to show that graphs belonging to those classes satisfy (1.1). We began the first part of this program in [9], but in order to complete it we need to understand the structure of reasonably well-connected cubic graphs that do not topologically contain the Petersen graph. That is the subject of [10], where we apply the structure theory of cyclically 5-connected cubic graphs developed in this paper. We have completed the second part of the project for doublecross graphs in [4]; the apex case is harder and is currently under preparation.

To motivate our structure theorems let us mention a special case of a theorem of Tutte [11].

**(1.2)** *Let  $G, H$  be non-isomorphic 3–connected cubic graphs, and let  $H$  contain  $G$  topologically. Then there exists a cubic graph  $G'$  obtained from  $G$  by subdividing two distinct edges of  $G$  and joining the new vertices by an edge in such a way such that  $H$  topologically contains  $G'$ .*

Our objective is to prove a similar theorem for cyclically 5-connected cubic graphs. An ideal analog of (1.2) for cyclically 5-connected cubic graphs would assert that there is a graph  $G'$  as in (1.2) that is cyclically 5-connected. That is unfortunately not true, but the exceptions can be conveniently described. We will do so now.

Let  $G$  be a cyclically 5-connected cubic graph. Let  $e, f$  be distinct edges of  $G$  with no common end and such that no edge of  $G$  is adjacent to both  $e$  and  $f$ , and let  $G'$  be obtained from  $G$  by subdividing  $e$  and  $f$  and joining the new vertices by an edge. We say that  $G'$  is a *handle expansion* of  $G$ . We show in (2.2) that  $G'$  is cyclically 5-connected. Let  $e_1, e_2, e_3, e_4, e_5$  (in order) be the edges of a circuit of  $G$  of length five. Let us subdivide

$e_i$  by a new vertex  $v_i$ , add a circuit (disjoint from  $G$ ) with vertices  $u_1, u_2, u_3, u_4, u_5$  (in order), and for  $i = 1, 2, \dots, 5$  let us add an edge joining  $u_i$  and  $v_i$  to form a graph  $G''$ . In these circumstances we say that  $G''$  is a *circuit expansion* of  $G$ . It is not hard to see, for instance by repeatedly applying (2.1), that  $G''$  is cyclically 5-connected.

Let  $p$  be an integer such that  $p \geq 5$  if  $p$  is odd and  $p \geq 10$  if  $p$  is even. Let  $G$  be a cubic graph with vertex-set  $\{u_0, u_1, \dots, u_{p-1}, v_0, v_1, \dots, v_{p-1}\}$  such that for  $i = 0, 1, \dots, p-1$ ,  $u_i$  has neighbors  $u_{i-1}$ ,  $u_{i+1}$  and  $v_i$ , and  $v_i$  has neighbors  $u_i$ ,  $v_{i-2}$  and  $v_{i+2}$ , where the index arithmetic is taken modulo  $p$  (see Figure 1). We say that  $G$  is a *biladder* on  $2p$  vertices. We remark that the Petersen graph is a biladder on 10 vertices, and that the Dodecahedron is a biladder on 20 vertices.

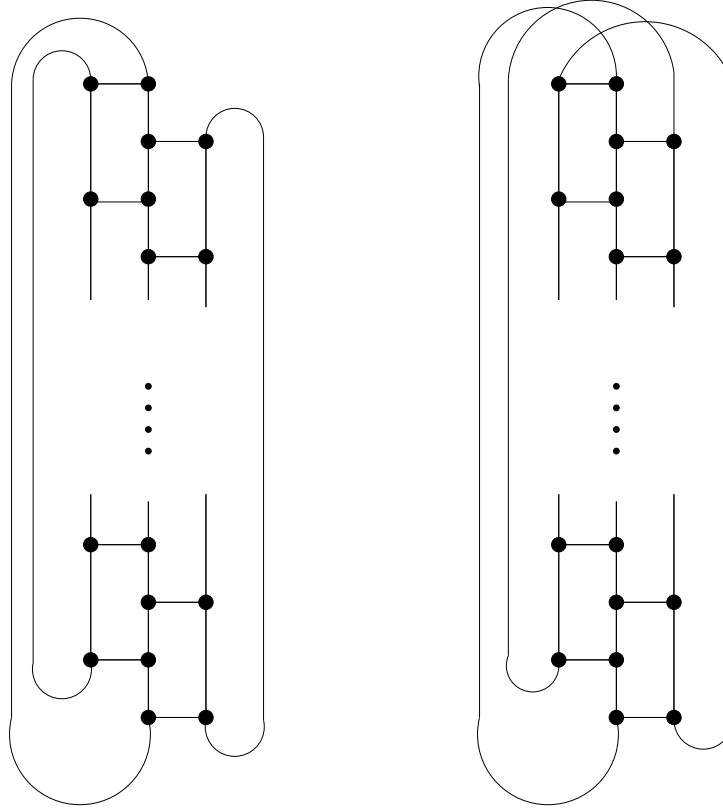


Figure 1: Biladders

The following is our first main result.

**(1.3)** *Let  $G, H$  be non-isomorphic cyclically 5-connected cubic graphs such that not both of them are biladders, let  $H$  contain  $G$  topologically, and assume that if  $G$  is isomorphic to the Petersen graph, then  $H$  does not topologically contain the biladder on 14 vertices, and if  $G$  is isomorphic to Dodecahedron, then  $H$  does not topologically contain the biladder on 24 vertices. Then there exists a cyclically 5-connected handle or circuit expansion  $G'$  of  $G$  such that  $H$  contains  $G'$  topologically.*

There is a variation of (1.3), which is easier to apply, but which involves a stronger assumption about the graph  $H$ . Dodecahedral connectivity is defined in Section 5.

**(1.4)** *Let  $G, H$  be non-isomorphic cyclically 5-connected cubic graphs such that not both of them are biladders, let  $H$  be dodecahedrally connected, let  $H$  contain  $G$  topologically, and assume that if  $G$  is isomorphic to the Petersen graph, then  $H$  does not topologically contain the biladder on 14 vertices, and if  $G$  is isomorphic to Dodecahedron, then  $H$  does not topologically contain the biladder on 24 vertices. Then there exists a cyclically 5-connected handle expansion  $G'$  of  $G$  such that  $H$  topologically contains  $G'$ .*

Since every biladder is either planar (if  $p$  is even), or topologically contains the Petersen graph (if  $p$  is odd) we deduce the following corollary.

**(1.5)** *Let  $G, H$  be non-isomorphic cyclically 5-connected cubic graphs, let  $G$  be non-planar, let  $H$  be dodecahedrally connected, let  $H$  contain  $G$  topologically, and assume that  $H$  does not topologically contain the Petersen graph. Then there exists a handle expansion  $G'$  of  $G$  such that  $H$  topologically contains  $G'$ .*

The last three theorems describe how to obtain a bigger cyclically 5-connected cubic graph from a smaller one. But what are the initial graphs to start from? The graphs Petersen, Triplex, Box, Ruby and Dodecahedron are defined in Figure 2. The following theorem of McCuaig [5, 6] was also obtained in [1].

**(1.6)** *Every cyclically 5-connected cubic graph topologically contains one of Petersen, Triplex, Box, Ruby or Dodecahedron.*

Theorems (1.3), (1.4) and (1.6) have the following corollary, the first part of which was proved for planar graphs in [2, 3], and for general graphs in [5, 6].

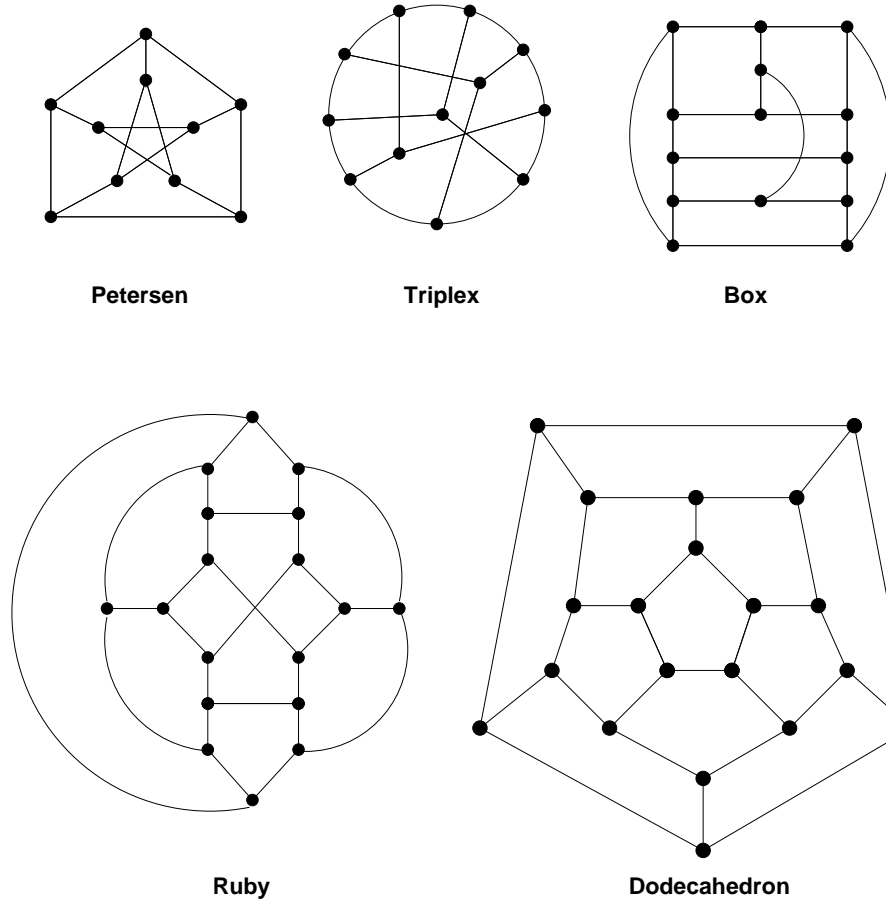


Figure 2

(1.7) *Every cyclically 5-connected cubic graph can be obtained from Triplex, Box, Ruby or a biladder by repeatedly applying the operations of handle expansion or circuit expansion. Every dodecahedrally connected cubic graph can be obtained from Triplex, Box, Ruby or a biladder by repeatedly applying the operation of handle expansion.*

It follows from (5.1) that a handle expansion of a dodecahedrally connected graph is again dodecahedrally connected.

Our proofs of (1.3) and (1.4) are indirect, and proceed by way of auxiliary results, stated as (4.10) and (5.6) below, that are themselves quite useful. Those auxiliary results allow  $G$  to violate the definition of cyclic 5-connectivity, but only in a limited way. For instance, if  $G$  satisfies the definition of cyclic 5-connectivity, except for one circuit of length four, then we can still insist that  $G'$  be cyclically 5-connected. However, the operations

that describe how to obtain  $G'$  are more complicated, and therefore we defer the exact statements to Section 4.

Let us introduce some terminology now. All graphs in this paper are finite and simple. Thus we may denote the edge of a graph with ends  $u$  and  $v$  by  $uv$  without any ambiguity. If  $G$  is a graph we denote its vertex-set and edge-set by  $V(G)$  and  $E(G)$ , respectively. Let  $G$  be a graph. If  $K, L$  are subgraphs of  $G$  we denote by  $K \cup L$  the graph with vertex-set  $V(K) \cup V(L)$ , edge-set  $E(K) \cup E(L)$  and the obvious incidences. If  $A \subseteq V(G)$  we denote by  $\delta_G A$  (or  $\delta A$  if the graph can be understood from the context) the set of edges of  $G$  with one end in  $A$  and the other end in  $V(G) - A$ . An *edge-cut* of  $G$  is a set of edges of the form  $\delta A$ , where  $A \subseteq V(G)$  and  $\emptyset \neq A \neq V(G)$ . If  $X$  is a vertex, a set of vertices, an edge, or a set of edges, we denote by  $G \setminus X$  the graph obtained from  $G$  by deleting  $X$ . If  $X$  is a set of vertices we denote by  $G|X$  the graph  $G \setminus (V(G) - X)$ . *Paths* and *circuits* have no “repeated” vertices and no “repeated” edges. A *quadrangle* is a circuit of length four. A graph  $G$  is *cubic* if every vertex of  $G$  has degree three and it is *subcubic* if every vertex has degree at most three. Let  $k \geq 4$  be an integer. We say that a cubic graph  $G$  is *cyclically  $k$ -connected* if  $G$  is 3-connected, has at least  $2k$  vertices, and for every edge-cut  $\delta A$  of  $G$  of cardinality less than  $k$ , one of  $G \setminus A$ ,  $G|A$  has no circuits.

Let  $e$  be an edge of a graph  $G$ . A graph  $H$  is obtained from  $G$  by *subdividing*  $e$  if  $H$  is obtained by deleting  $e$ , adding a new vertex  $v \notin V(G)$ , and joining  $v$  to both ends of  $e$  by new edges. We say that  $v$  is the *new vertex* of  $H$ . We say that a graph  $H$  *topologically contains* a graph  $G$  if some graph obtained from  $G$  by repeatedly subdividing edges is isomorphic to a subgraph of  $H$ .

The paper is organized as follows. In Sections 2 and 3 we introduce some terminology and prove several lemmas. In Section 4 we solve the following problem: Suppose that a cyclically 5-connected cubic graph  $H$  contains a graph  $G$  topologically and is minimal with this property, where  $G$  is “almost” cyclically 5-connected (quad-connected, as defined in the next section). What can we say about  $H$ ? In Section 5 we strengthen the conclusion of the result of Section 4 under the assumption that  $H$  is dodecahedrally connected. In Section 6 we prove a preliminary version of (1.3), where we allow adding two handles, rather than one. We prove (1.3) and (1.4) in Section 7.

## 2. EXTENSIONS

Let  $G$  be a cubic graph. We say that  $G$  is *quad-connected* if

- $G$  is cyclically 4-connected,
- $G$  has at least 10 vertices,
- if  $G$  has more than one quadrangle, then it has at least 12 vertices, and
- for every edge-cut  $\delta A$  of  $G$  of cardinality at most four, one of  $G|A$  and  $G \setminus A$  is a forest or a quadrangle.

Thus a cubic graph is cyclically 5-connected if and only if it is quad-connected and has no quadrangle.

Let  $u, v, x, y$  be vertices of a graph  $G$  such that  $u$  is adjacent to  $v$ ,  $x$  is adjacent to  $y$ , and  $\{u, v\} \neq \{x, y\}$ . We define  $G + (u, v, x, y)$  to be the graph obtained from  $G$  by subdividing the edges  $uv$  and  $xy$ , where the new vertices are  $k$  and  $l$ , respectively, and adding an edge joining  $k$  and  $l$ . The vertices  $k, l$  (in this order) will be called the *new vertices* of  $G + (u, v, x, y)$ . We remark that if  $u, v, x, y$  are pairwise distinct and  $G$  has no circuits of length at most three, then neither does  $G + (u, v, x, y)$ . If the vertices  $u, v, x, y$  are pairwise distinct, then we say that  $G + (u, v, x, y)$  is a *1-extension* of  $G$ . If, in addition, neither  $u$  nor  $v$  is adjacent to  $x$  or  $y$ , then we say that  $G + (u, v, x, y)$  is a *long 1-extension* of  $G$ ; otherwise we say that it is *short*. Thus if  $G$  is cyclically 5-connected, then long 1-extension and handle expansion mean the same thing.

Now let  $C$  be a quadrangle in  $G$ . We say that the 1-extension  $G + (u, v, x, y)$  is *based at  $C$*  if  $uv$  is an edge of  $C$  and  $x, y \notin V(C)$ , and we will apply the qualifiers long and short as in the previous paragraph. Let  $G$  be quad-connected, let  $C$  be a quadrangle in  $G$ , let  $G + (u, v, x, y)$  be a short 1-extension of  $G$  based at  $C$ , and let  $k, l$  be the new vertices. Then one of  $u, v$  is adjacent to one of  $x, y$ , and so we may assume that say  $u$  is adjacent to  $x$ . Then  $\{u, x, k, l\}$  is the vertex-set of a quadrangle  $D$  in  $G'$ . The next lemma implies that  $D$  is the only quadrangle of  $G'$  containing the edge  $kl$ . We say that  $D$  is the *new quadrangle* of  $G'$ .

**(2.1)** *Let  $G$  be a quad-connected graph, and let  $G'$  be a 1-extension of  $G$  such that if  $G$  has a quadrangle, then  $G'$  is a 1-extension of  $G$  based at some quadrangle of  $G$ . Then  $G'$*



is quad-connected. In particular,  $G'$  has at most one quadrangle that is not a quadrangle of  $G$ .

*Proof.* The second assertion follows from the first. To prove that  $G'$  is quad-connected it suffices to verify the last condition in the definition of quad-connectivity, because the other conditions are clear. The graph  $G'$  is clearly 3-connected. Let  $k, l$  be the new vertices of  $G'$ . Let  $\delta_{G'}A$  be an edge-cut of  $G'$  of cardinality at most four such that both  $G'|A$  and  $G' \setminus A$  contain circuits. We must show that  $|\delta_{G'}A| = 4$  and that  $G'|A$  or  $G' \setminus A$  is a quadrangle. We have  $4 \leq |A| \leq |V(G')| - 4$ . Let  $B = A - \{k, l\}$ . Then  $\emptyset \neq B \neq V(G)$ , and so  $\delta_GB$  is an edge-cut of  $G$  of cardinality at most four. Thus one of  $G|B$  and  $G \setminus B$  is a forest or a quadrangle.

Suppose first that  $G|B$  is a quadrangle. Since  $4 = |\delta_GB| \leq |\delta_{G'}A| \leq 4$ , we see that  $\delta_GB = \delta_{G'}A$ . The definition of 1-extension implies that  $\{u, v\} \not\subseteq B$  and  $\{x, y\} \not\subseteq B$ . Thus  $G|B = G'|A$ , and so  $G'|A$  is a quadrangle, as desired. This completes the case where  $G|B$  is a quadrangle.

By symmetry between  $G|B$  and  $G \setminus B$  we may therefore assume that  $G|B$  is a forest. Since  $|\delta_GB| \leq 4$  we see that  $|B| \leq 2$ , and since  $|A| \geq 4$  we have  $|B| = 2$ , say  $B = \{a, b\}$ . Thus  $A = \{a, b, k, l\}$ ,  $|\delta_{G'}A| = 4$ , and  $G'|A$  is a quadrangle, as required.  $\square$

**(2.2)** *Let  $G$  be a quad-connected cubic graph with at most one quadrangle, and let  $G'$  be a long 1-extension such that if  $G$  has a quadrangle  $C$ , then  $G'$  is a 1-extension based at  $C$ . Then  $G'$  is cyclically 5-connected.*

*Proof.* The graph  $G'$  is quad-connected by (2.1). Since the extension is long, the graph  $G'$  has no quadrangle, and hence is cyclically 5-connected.  $\square$

**(2.3)** *Let  $G$  be a quad-connected cubic graph, let the vertices  $u_1, u_2, u_3, u_4, u_5$  (in order) form the vertex-set of a path of  $G$ , let  $G' = G + (u_1, u_2, u_4, u_5)$ , and assume that either  $G$  is cyclically 5-connected, or  $G$  has a quadrangle  $C$  with  $u_1, u_2 \in V(C)$  and  $u_4, u_5 \notin V(C)$ . Then  $G'$  is a short extension of  $G$  if and only if  $u_1$  and  $u_5$  are adjacent.*

*Proof.* If  $u_1$  and  $u_5$  are adjacent, then  $G'$  is clearly a short extension. Conversely, if  $G'$  is a short extension, then one of  $u_1, u_2$  is adjacent to one of  $u_4, u_5$ . Since  $G$  has no triangles

we may assume for a contradiction that either  $u_1$  is adjacent to  $u_4$ , or  $u_2$  is adjacent to  $u_5$ . In either case  $G$  has a quadrangle  $D \neq C$ , and hence  $G$  is not cyclically 5-connected. Thus  $C$  exists, but the existence of  $C$  and  $D$  contradicts the quad-connectivity of  $G$ .  $\square$

Let  $G$  be a cyclically 4-connected cubic graph, let  $u_1, u_2, \dots, u_6$  be the vertices of a path in  $G$  in order, let  $G_1 = G + (u_1, u_2, u_3, u_4)$ , and let  $k_1, l_1$  be the new vertices of  $G_1$ . We define  $G \& (u_1, u_2, u_3, u_4, u_5, u_6)$  to be the graph  $G_2 = G_1 + (u_3, l_1, u_5, u_6)$ . Let  $k_2, l_2$  be the new vertices of  $G_2$ . We say that  $k_1, l_1, k_2, l_2$  are the *new vertices* of  $G \& (u_1, u_2, u_3, u_4, u_5, u_6)$ .

**(2.4)** *Let  $G$  be a cubic graph, and let  $u_1, u_2, \dots, u_6$  be vertices of  $G$  forming the vertex-set of a path in the order listed. Let  $G' = G \& (u_1, u_2, u_3, u_4, u_5, u_6)$ . Assume that  $G$  is quad-connected with at most one quadrangle, and that if it has a quadrangle, then it has a quadrangle  $C$  with  $u_1, u_2 \in V(C)$  and  $u_4, u_5, u_6 \notin V(C)$ . Then  $G'$  is cyclically 5-connected.*

*Proof.* By (2.1)  $G_1 = G + (u_1, u_2, u_3, u_4)$  is quad-connected, and it has exactly one quadrangle. By another application of (2.1) the graph  $G'$  is cyclically 5-connected, because it has no quadrangle.  $\square$

**(2.5)** *Let  $G$  be a cubic graph, let  $u_1, u_2, \dots, u_5$  be the vertices of a circuit of  $G$  in order, and assume that  $G$  is either cyclically 5-connected, or quad-connected with a quadrangle  $C$  such that  $u_3, u_4 \in V(C)$  and  $u_1, u_2, u_5 \notin V(C)$ . Let  $v_1$  be the neighbor of  $u_1$  other than  $u_2$  and  $u_5$ , and let  $G' = G + (u_3, u_4, u_1, v_1)$ . Then  $G'$  is a long 1-extension of  $G$ .*

*Proof.* The vertex  $u_3$  is not adjacent to  $v_1$  in  $G$ , for otherwise  $G$  has a quadrangle  $D$  with vertex-set  $\{v_1, u_1, u_2, u_3\}$ , which implies that  $C$  exists, but the existence of  $C$  and  $D$  contradicts the quad-connectivity of  $G$ . Hence  $G'$  is a long 1-extension by (2.3).  $\square$

**(2.6)** *Let  $G$  be a quad-connected cubic graph, and let  $C$  be a quadrangle in  $G$  with vertices  $u_1, u_2, u_3, u_4$  in order. Let  $v_1$  be the neighbor of  $u_1$  not on  $C$ , and let  $v_2$  be defined similarly. Let  $w_1 \neq u_1$  be a neighbor of  $v_1$ , and let  $z_1 \neq v_1$  be a neighbor of  $w_1$ . Then  $G + (u_2, u_3, v_1, w_1)$  is a long 1-extension of  $G$ , and if  $z_1 \neq v_2$  then  $G + (u_1, u_2, w_1, z_1)$  is a long 1-extension of  $G$ .*

*Proof.* The vertices  $w_1$  and  $u_3$  are not adjacent, for otherwise the set  $\{u_1, u_2, u_3, u_4, v_1, w_1\}$  contradicts the quad-connectivity of  $G$ . Thus the 1-extension  $G + (u_3, u_4, v_1, w_1)$  is long by (2.3), and so is  $G + (u_1, u_2, w_1, z_1)$  if  $z_1 \neq v_2$ .  $\square$

### 3. HOMEOMORPHIC EMBEDDINGS

Let  $G, H$  be graphs. A mapping  $\eta$  with domain  $V(G) \cup E(G)$  is called a *homeomorphic embedding* of  $G$  into  $H$  if for every two vertices  $v, v'$  and every two edges  $e, e'$  of  $G$

- (i)  $\eta(v)$  is a vertex of  $H$ , and if  $v, v'$  are distinct then  $\eta(v), \eta(v')$  are distinct,
- (ii) if  $e$  has ends  $v, v'$ , then  $\eta(e)$  is a path of  $H$  with ends  $\eta(v), \eta(v')$ , and otherwise disjoint from  $\eta(V(G))$ , and
- (iii) if  $e, e'$  are distinct, then  $\eta(e)$  and  $\eta(e')$  are edge-disjoint, and if they have a vertex in common, then this vertex is an end of both.

We shall denote the fact that  $\eta$  is a homeomorphic embedding of  $G$  into  $H$  by writing  $\eta : G \hookrightarrow H$ , and we shall write  $G \hookrightarrow H$  to mean that there exists a homeomorphic embedding of  $G$  into  $H$ . If  $K$  is a subgraph of  $G$  we denote by  $\eta(K)$  the subgraph of  $H$  consisting of all vertices  $\eta(v)$ , where  $v \in V(K)$ , and all vertices and edges that belong to  $\eta(e)$  for some  $e \in E(K)$ . It is easy to see that  $H$  contains  $G$  topologically if and only if there is a homeomorphic embedding  $G \hookrightarrow H$ .

Let  $G_0$  be a quad-connected graph, let  $C_0$  be a quadrangle in  $G$ , and let  $n \geq 1$  be an integer. We say that  $G_n$  is an *n-extension* of  $G_0$  *based at*  $C_0$  if there exists a sequence  $G_1, C_1, G_2, C_2, \dots, G_n$  such that for  $i = 1, 2, \dots, n$ , the graph  $G_i$  is a 1-extension of  $G_{i-1}$  based at  $C_{i-1}$ , and if  $i < n$  then this 1-extension is short and  $C_i$  is the new quadrangle in  $G_i$ . We say that  $G_n$  is a *short n-extension* of  $G_0$  if  $G_n$  is a short 1-extension of  $G_{n-1}$ , and we say that it is a *long n-extension* otherwise. We say that the sequence  $G_1, G_2, \dots, G_n$  is a *generating sequence of the n-extension  $G_n$  from  $G_0$  based at  $C_0$* . We say that a graph  $H$  is an *extension* of  $G_0$  if it is an *n-extension* for some integer  $n \geq 1$ .

If  $G_0, G_1, \dots, G_n$  are as in the previous paragraph, then for each  $i = 1, 2, \dots, n$  there is a natural homeomorphic embedding  $G_{i-1} \hookrightarrow G_i$ , and hence there is a natural homeomorphic embedding  $\iota : G_0 \hookrightarrow G_n$ , called the *canonical embedding determined by the*

generating sequence  $G_1, G_2, \dots, G_n$ . When there is no danger of confusion we will drop the reference to the generating sequence and simply talk about a canonical embedding.

Let  $G, H, K$  be graphs, and let  $\eta : G \hookrightarrow H$  and  $\zeta : H \hookrightarrow K$ . For  $v \in V(G)$  we put  $\xi(v) = \zeta(\eta(v))$ , and for  $e \in E(G)$  we define  $\xi(e)$  to be the union of  $\zeta(f)$  over all edges  $f \in E(\eta(e))$ . Then  $\xi : G \hookrightarrow K$ , and we write  $\xi = \eta \circ \zeta$ .

Frequently will need to construct new homeomorphic embeddings from old ones by means of “rerouting”. We now introduce these constructions formally. Let  $G, H$  be graphs, and let  $\eta : G \hookrightarrow H$  be a homeomorphic embedding. Let  $e \in E(G)$ , and let  $P'$  be a path in  $H$  of length at least one with both ends on  $\eta(e)$ , and otherwise disjoint from  $\eta(G)$ . Let  $P$  be the subpath of  $\eta(e)$  with ends the ends of  $P'$ . Let  $\eta'(e)$  be the path obtained from  $\eta(e)$  by replacing the interior of  $P$  by  $P'$ , and let  $\eta'(x) = \eta(x)$  for all  $x \in V(G) \cup E(G) - \{e\}$ . Then  $\eta' : G \hookrightarrow H$  is a homeomorphic embedding, and we say that  $\eta'$  was obtained from  $\eta$  by *rerouting  $\eta(e)$  along  $P'$* .

Let  $e, f, g$  be three distinct edges of  $G$ , all incident with a vertex  $v$  of degree three. Let  $x$  be an interior vertex of  $\eta(e)$ , let  $y$  be an interior vertex of  $\eta(f)$ , and let  $P'$  be a path in  $H$  with ends  $x$  and  $y$ , and otherwise disjoint from  $\eta(G)$ . Let  $\eta'(v) = y$ , let  $\eta'(e)$  be obtained from  $\eta(e)$  by deleting the part from  $x$  to  $\eta(v)$  (including  $\eta(v)$  but not  $x$ ) and adding  $P'$ , let  $\eta'(f)$  be obtained from  $\eta(f)$  by deleting the part from  $y$  to  $\eta(v)$  (including  $\eta(v)$  but not  $y$ ), and let  $\eta'(g)$  be obtained from  $\eta(g)$  by adding the subpath of  $\eta(f)$  with ends  $y$  and  $\eta(v)$ . For  $x \in V(G) \cup E(G) - \{v, e, f, g\}$  let  $\eta'(x) = \eta(x)$ . Then  $\eta' : G \hookrightarrow H$ , and we say that  $\eta'$  was obtained from  $\eta$  by *rerouting  $\eta(e)$  along  $P'$* .

Let  $e$  be an edge of  $G$  with ends  $u, v$  of degree three, let  $f_1, f_2$  be the other two edges incident with  $u$ , and let  $g_1, g_2$  be the other two edges incident with  $v$ . Let  $x$  be an interior vertex of  $\eta(f_1)$ , let  $y$  be an interior vertex of  $\eta(g_1)$ , and let  $P'$  be a path in  $H$  with ends  $x$  and  $y$ , and otherwise disjoint from  $\eta(G)$ . Let  $\eta'(u) = x$ , let  $\eta'(v) = y$ , let  $\eta'(e) = P'$ , let  $\eta'(f_1)$  be the path obtained from  $\eta(f_1)$  by deleting the subpath between  $x$  and  $\eta(u)$  (including  $\eta(u)$  but not  $x$ ), let  $\eta'(g_1)$  be the path obtained from  $\eta(g_1)$  by deleting the subpath between  $y$  and  $\eta(v)$  (including  $\eta(v)$  but not  $y$ ), let  $\eta'(f_2)$  be obtained from  $\eta(f_2)$  by adding the subpath of  $\eta(f_1)$  between  $x$  and  $\eta(u)$ , and let  $\eta'(g_2)$  be obtained from  $\eta(g_2)$  by adding the subpath of  $\eta(g_1)$  with ends  $y$  and  $\eta(v)$ . For  $x \in V(G) \cup E(G) - \{u, v, e, f_1, f_2, g_1, g_2\}$  let

$\eta'(x) = \eta(x)$ . Then  $\eta' : G \hookrightarrow H$ , and we say that  $\eta'$  was obtained from  $\eta$  by rerouting  $\eta(e)$  along  $P'$ .

Our next objective is to analyze augmenting paths relative to homeomorphic embeddings. The next lemma follows by a standard application of network flow theory.

**(3.1)** *Let  $k \geq 0$  be an integer, let  $G, H$  be cubic graphs, let  $\delta_G A = \{e_1, e_2, \dots, e_k\}$  be an edge-cut of  $G$  of cardinality  $k$ , and for  $i = 1, 2, \dots, k$  let the ends of  $e_i$  be  $u_i \in A$  and  $v_i \in V(G) - A$ . Let  $\eta : G \hookrightarrow H$  be a homeomorphic embedding, and assume that there is no edge-cut  $\delta_H B$  of  $H$  of cardinality  $k$  with  $\eta(A) \subseteq B$  and  $\eta(V(G) - A) \subseteq V(H) - B$ . Then there exist an integer  $n$  and disjoint paths  $Q_1, Q_2, \dots, Q_n$  in  $H$ , where  $Q_i$  has distinct ends  $x_i$  and  $y_i$  such that*

- (i)  $x_1 \in V(\eta(G|A)) - \{\eta(u_1), \eta(u_2), \dots, \eta(u_k)\}$  and  $y_n \in V(\eta(G \setminus A)) - \{\eta(v_1), \eta(v_2), \dots, \eta(v_k)\}$ ,
- (ii) for all integers  $i \in \{1, 2, \dots, n-1\}$ , the vertices  $x_{i+1}, y_i \in V(\eta(e_t))$  for some  $t \in \{1, 2, \dots, k\}$ , and  $\eta(u_t), x_{i+1}, y_i, \eta(v_t)$  are pairwise distinct and occur on  $\eta(e_t)$  in the order listed,
- (iii) if  $x_i, y_j \in V(\eta(e_t))$  for some  $t \in \{1, 2, \dots, k\}$  and  $i, j \in \{1, 2, \dots, n\}$  with  $i > j + 1$ , then  $\eta(u_t), y_j, x_i, \eta(v_t)$  occur on  $\eta(e_t)$  in the order listed, and
- (iv) for  $i = 1, 2, \dots, n$ , if a vertex of  $Q_i$  belongs to  $V(\eta(G))$ , then it is an end of  $Q_i$ .

In the situation described in (3.1) we call the sequence of paths  $\gamma = (Q_1, Q_2, \dots, Q_n)$  an *augmenting sequence with respect to  $G, H, A$  and  $\eta$* . Let  $F$  be a graph. We say that  $\gamma$  is *reduced modulo  $F$*  if the following conditions are satisfied:

- (i) If  $e \in E(G|A)$  and  $t \in \{1, 2, \dots, k\}$  are such that  $x_1 \in V(\eta(e))$  and  $y_1 \in V(\eta(e_t))$ , then  $e$  and  $e_t$  have no common end, and no end of  $e$  is adjacent to an end of  $e_t$  in  $G \setminus E(F)$ .
- (ii) If  $t \in \{1, 2, \dots, k\}$  and  $f \in E(G \setminus A)$  are such that  $x_n \in V(\eta(e_t))$  and  $y_n \in V(\eta(f))$ , then  $e_t$  and  $f$  have no common end, and no end of  $e_t$  is adjacent to an end of  $f$  in  $G \setminus E(F)$ .
- (iii) If  $t, t' \in \{1, 2, \dots, k\}$  and  $i \in \{2, 3, \dots, n-1\}$  are such that  $x_i \in V(\eta(e_t))$  and  $y_i \in V(\eta(e_{t'}))$ , then  $t \neq t'$ ,  $u_t$  is not adjacent to  $u_{t'}$  in  $G \setminus E(F)$ , and  $v_t$  is not adjacent

to  $v_{t'}$  in  $G \setminus E(F)$ .

Let  $G, H$  be graphs, let  $\eta : G \hookrightarrow H$ , and let  $F$  be a graph of minimum degree at least two. We say that the homeomorphic embedding  $\eta$  *fixes*  $F$  if  $F$  is a subgraph of both  $G$  and  $H$ ,  $\eta(v) = v$  for every vertex  $v \in V(F)$  and for every edge  $e \in E(F)$  the image  $\eta(e)$  is the path with edge-set  $\{e\}$ . In many of our lemmas and theorems we will be able to find a homeomorphic embedding that fixes a specified graph  $F$ . This feature will not be needed in this or the follow-up paper [10], but is included because it may be useful in future applications. As far as this paper and [10] are concerned, the reader may take  $F$  to be the null graph.

The lemma we need is the following.

**(3.2)** *Let  $G, H$  be cubic graphs, let  $\delta_G A$  be an edge-cut in  $G$  such that no two members of  $\delta_G A$  have a common end, let  $F$  be a graph of minimum degree at least two, let  $\eta : G \hookrightarrow H$  be a homeomorphic embedding that fixes  $F$ , and let  $\gamma$  be an augmenting sequence with respect to  $G, H, A$  and  $\eta$  of length  $n$ . Let us assume that  $\gamma$  is minimal in the sense that there is no homeomorphic embedding  $\eta' : G \hookrightarrow H$  that fixes  $F$  and an augmenting sequence with respect to  $G, H, A$  and  $\eta'$  of length  $n'$  such that  $n' < n$ . Then  $\gamma$  is reduced modulo  $F$ .*

*Proof.* Let  $G, H, A, \eta$  and  $\gamma$  and be as stated, let  $\delta_G A = \{e_1, e_2, \dots, e_k\}$ , and let  $\gamma = (Q_1, Q_2, \dots, Q_n)$ . To prove that  $\gamma$  satisfies (i) let  $t$  and  $e$  be as in (i), and suppose for a contradiction that either  $e$  and  $e_t$  have a common end, or that some end of  $e$  is adjacent to some end of  $e_t$  in  $G \setminus E(F)$ . Since  $G$  and  $H$  are cubic,  $x_1$  is an interior vertex of  $\eta(e)$  and  $y_1$  is an interior vertex of  $\eta(e_t)$ . Let  $\eta'$  be obtained from  $\eta$  by rerouting  $\eta(e)$  (if  $e$  and  $f$  have a common end) or  $\eta(g)$  (where  $g$  is an edge of  $G \setminus E(F)$  adjacent to both  $e$  and  $f$ ) along  $Q_1$ . Then  $Q_2, Q_3, Q_4, \dots, Q_n$  is an augmenting sequence with respect to  $G, H, A$  and  $\eta'$ , and hence  $\gamma$  is not minimal, a contradiction.

Condition (ii) follows similarly, and so it remains to prove (iii). To that end let  $t, t'$  and  $i$  be as in (iii). Suppose first that  $t = t'$ . Then  $\eta(u_t), x_i, y_{i-1}, x_{i+1}, y_i, \eta(v_t)$  all belong to  $\eta(e_t)$  and occur on  $\eta(e_t)$  in the order listed. Let  $\eta'$  be obtained from  $\eta$  by rerouting  $\eta(e_t)$  along  $Q_i$ , and let  $Q$  be the union of  $Q_{i-1}$ ,  $Q_{i+1}$  and the subpath of  $\eta(e_t)$  with ends  $y_{i-1}$  and  $x_{i+1}$ . Then  $Q_1, Q_2, \dots, Q_{i-2}, Q, Q_{i+2}, \dots, Q_n$  is an augmenting sequence with

respect to  $G, H, A$  and  $\eta'$ , and hence  $\gamma$  is not minimal, a contradiction.

Thus  $t \neq t'$ . Next we suppose for a contradiction that  $u_t$  is adjacent to  $u_{t'}$  in  $G \setminus E(F)$ . Let  $\eta'$  be obtained from  $\eta$  by rerouting  $\eta(u_t u_{t'})$  along  $Q_i$ ; then  $Q_{i+1}, Q_{i+2}, \dots, Q_n$  is an augmenting sequence with respect to  $G, H, A$  and  $\eta'$ , and hence  $\gamma$  is not minimal, a contradiction. Similarly we deduce that  $v_t$  is not adjacent to  $v_{t'}$ . Thus  $\gamma$  is reduced, as required.  $\square$

Let  $G, H$  be cubic graphs, let  $\eta : G \hookrightarrow H$  be a homeomorphic embedding, let  $e_1, e_2$  be two edges of  $G$  with ends  $u_1, v_1$  and  $u_2, v_2$ , respectively, where  $u_1, v_1, u_2, v_2$  are pairwise distinct, and assume that there exists a path  $Q$  in  $H$  with ends  $x_i \in V(\eta(e_i))$  ( $i = 1, 2$ ) and otherwise disjoint from  $\eta(G)$ . Let  $G' = G + (u_1, v_1, u_2, v_2)$ , and let  $k_1, k_2$  be the new vertices of  $G'$ ; then  $G'$  is a 1-extension of  $G$ . For  $i = 1, 2$  let  $\eta'(k_i) = x_i$ , let  $\eta'(k_1 k_2) = Q$ , let  $\eta'(u_i k_i)$  be the subpath of  $\eta(u_i v_i)$  with ends  $\eta(u_i), x_i$ , let  $\eta'(v_i k_i)$  be defined analogously, and let  $\eta'(x) = \eta(x)$  for all  $x \in V(G) \cup E(G) - \{e_1, e_2\}$ . Then  $\eta' : G' \hookrightarrow H$  is a homeomorphic embedding. We say that the pair  $G', \eta'$  was obtained from  $\eta$  by *routing the new edge along  $Q$* .

**(3.3)** *Let  $G$  be a cubic graph, let  $H$  be a cyclically 5-connected cubic graph, let  $F$  be a graph of minimum degree at least two, let  $\eta : G \hookrightarrow H$  fix  $F$ , let  $C$  be a quadrangle in  $G$  that is disjoint from  $F$ , and assume that  $G$  has a circuit disjoint from  $C$ . Then there exist a 1-extension  $G'$  of  $G$  based at  $C$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow H$  that fixes  $F$ .*

*Proof.* Since  $H$  is cyclically 5-connected and  $G \setminus V(C)$  contains a circuit, by (3.1) there exists an augmenting sequence  $\gamma = (Q_1, Q_2, \dots, Q_n)$  with respect to  $G, H, V(C)$  and  $\eta$ . By (3.2) we may assume that  $\gamma$  is reduced modulo  $F$ . Let  $G', \eta'$  be obtained from  $\eta$  by routing the new edge along  $Q_1$ . Then  $G', \eta'$  satisfy the conclusion of the lemma.  $\square$

**(3.4)** *Let  $G, H$  be non-isomorphic cubic graphs, let  $F$  be a graph of minimum degree at least two, let  $\eta : G \hookrightarrow H$  fix  $F$ , and let  $G$  and  $H$  be cyclically 4-connected. Then there exist a 1-extension  $G'$  of  $G$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow H$  such that  $\eta'$  fixes  $F$ .*

*Proof.* Since  $G$  is not isomorphic to  $H$ , and  $H$  is cyclically 4-connected, there exists a path  $P$  in  $H$  with at least one edge, with both ends on  $\eta(G)$ , and otherwise disjoint from  $\eta(G)$ . Let  $x_1 \in \eta(e_1)$  and  $x_2 \in \eta(e_2)$  be the ends of  $P$ , where  $e_1, e_2 \in E(G)$ . Let  $u, u_1$  be the ends of  $e_1$ , and let  $v, u_2$  be the ends of  $e_2$ . If  $u, u_1, v, u_2$  are pairwise distinct, then  $G' = G + (u, u_1, v, u_2)$  and  $\eta'$  obtained from  $\eta$  by routing the new edge along  $P$  satisfy the conclusion of the lemma. We may therefore assume that say  $u = v$ . The case when  $u_1 = u_2$  can be reduced to the case  $u_1 \neq u_2$  by a similar, though easier argument, and is omitted. Thus we assume that  $u = v$  and  $u_1 \neq u_2$ .

Let  $G_1$  be obtained from  $G$  by subdividing  $e_1$  and  $e_2$  and joining the new vertices by an edge. Let  $v_1$  and  $v_2$  be the new vertices of  $G_1$  numbered so that  $v_1$  resulted by subdividing  $e_1$ . Let  $\eta_1 : G_1 \hookrightarrow H$  be obtained by routing the new edge along  $P$ , and let  $A = \{u, v_1, v_2\}$ . Since  $H$  is cyclically 4-connected, there exists, by (3.1), an augmenting sequence with respect to  $G_1, H, A$  and  $\eta_1$ . By (3.2) we may assume, by replacing  $\eta_1$  by a different homeomorphic embedding if necessary, that there exists a path  $Q_1$  as in (3.2). Let  $x \in V(\eta_1(G_1|A))$  and  $y \in V(\eta_1(G_1 \setminus A))$  be the ends of  $Q_1$ ; let  $f \in E(G \setminus A)$  be such that  $y \in V(\eta_1(f))$ . From the symmetry between  $e_1$  and  $e_2$  we may assume that  $x$  belongs to  $\eta(e_1) \cup \eta_1(v_1 v_2)$ . Thus  $P \cup Q_1$  has a subpath  $R$  with one end in  $\eta(e_1)$ , the other end  $y$  and otherwise disjoint from  $\eta(G)$ . If  $f$  is not incident with  $u_1$ , then the graph and homeomorphic embedding obtained from  $G$  by routing the new edge along  $R$  are as desired. Thus we may assume that  $f$  is incident with  $u_1$ . Let  $\eta' : G \hookrightarrow H$  be obtained from  $G$  by rerouting  $\eta(e_2)[u, v_2]$  along  $P$ ; then the graph and homeomorphic embedding obtained from  $\eta'$  by routing the new edge along  $Q_1$  are as desired.  $\square$

#### 4. FIXING A QUADRANGLE

Let  $G$  be a quad-connected cubic graph, and let  $C$  be a quadrangle in  $G$ . In this section we study the following problem: If  $H$  is cyclically 5-connected and topologically contains  $G$ , is there a quad-connected cubic graph  $G'$  such that  $G'$  is obtained from  $G$  by one of a set of well-defined operations,  $G$  is topologically contained in  $G'$ ,  $G'$  is topologically contained in  $H$  and has fewer quadrangles than  $G$ ? The following simple result gives a preliminary answer. Let us recall that extensions were defined at the beginning of Section 3.



(4.1) Let  $G$  be a quad-connected cubic graph, let  $H$  be a cyclically 5-connected cubic graph, let  $F$  be a graph of minimum degree at least two, and let  $\eta : G \hookrightarrow H$  fix  $F$ . Let  $C$  be a quadrangle in  $G$  that is disjoint from  $F$ . Then there exist an integer  $n \geq 1$ , a long  $n$ -extension  $G'$  of  $G$  based at  $C$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow H$  that fixes  $F$ .

*Proof.* Let  $n$  be the maximum integer such that there exists an  $n$ -extension  $G'$  of  $G$  based at  $C$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow H$  that fixes  $F$ . This is well-defined, because  $2k \leq |V(H)| - |V(G)|$  for every  $k$ -extension of  $G$ . We claim that  $G'$  is long. To prove the claim suppose to the contrary that it is short, and let  $C'$  be the new quadrangle of  $G'$ . It follows that  $C'$  is disjoint from  $F$ . By (3.3) there exists a 1-extension  $G''$  of  $G'$  based at  $C'$  and a homeomorphic embedding  $\eta'' : G'' \hookrightarrow H$  that fixes  $F$ . Then  $G''$  is an  $(n+1)$ -extension of  $G$  based at  $C$ , contrary to the choice of  $n$ . This proves our claim that  $G'$  is a long extension of  $G$ , and hence the pair  $G', \eta'$  satisfies the conclusion of the lemma.  $\square$

In the rest of this section we strengthen (4.1) in two ways: we give a bound on the minimum integer  $n$  that satisfies the conclusion of (4.1), and we give an explicit list of long extensions based at  $C$  such that one of them is guaranteed to satisfy (4.1). We now introduce these extensions.

Let  $G$  be a quad-connected graph, let  $C$  be a cycle in  $G$ , let  $u_1, u_2, u_3, u_4$  be the vertices of  $C$  in order, for  $i = 1, 2, 3, 4$  let  $v_i$  be the unique neighbor of  $u_i$  not on  $C$ , and let  $v'_i \neq u_i$  be a neighbor of  $v_i$ . It follows that  $v'_i \notin \{v_1, v_2, v_3, v_4\}$ . Let  $G_1 = G + (u_1, u_2, x, y)$  be a 1-extension of  $G$  with  $x, y \notin V(C)$ .

- If  $G_1$  is a long extension of  $G$ , we say that  $G_1$  is a *type A expansion* of  $G$  based at  $C$ . See Figures 3, 4 and 5. Otherwise we may assume that say  $x = v_1$  and  $y = v'_1$ . Let  $C_1$  be the new quadrangle of  $G_1$ ; thus  $C_1$  has vertex-set  $\{v_1, u_1, k, l\}$ , where  $k, l$  are the new vertices of  $G_1$ .

- Let  $G_2 = G_1 + (v_1, l, a, b)$  be a 1-extension of  $G_1$ . If  $G_2$  is a long extension of  $G$  we say that  $G_2$  is a *type B expansion* of  $G$  based at  $C$ , and that the sequence  $G_1, G_2$  is a *standard generating sequence* of  $G_2$ .

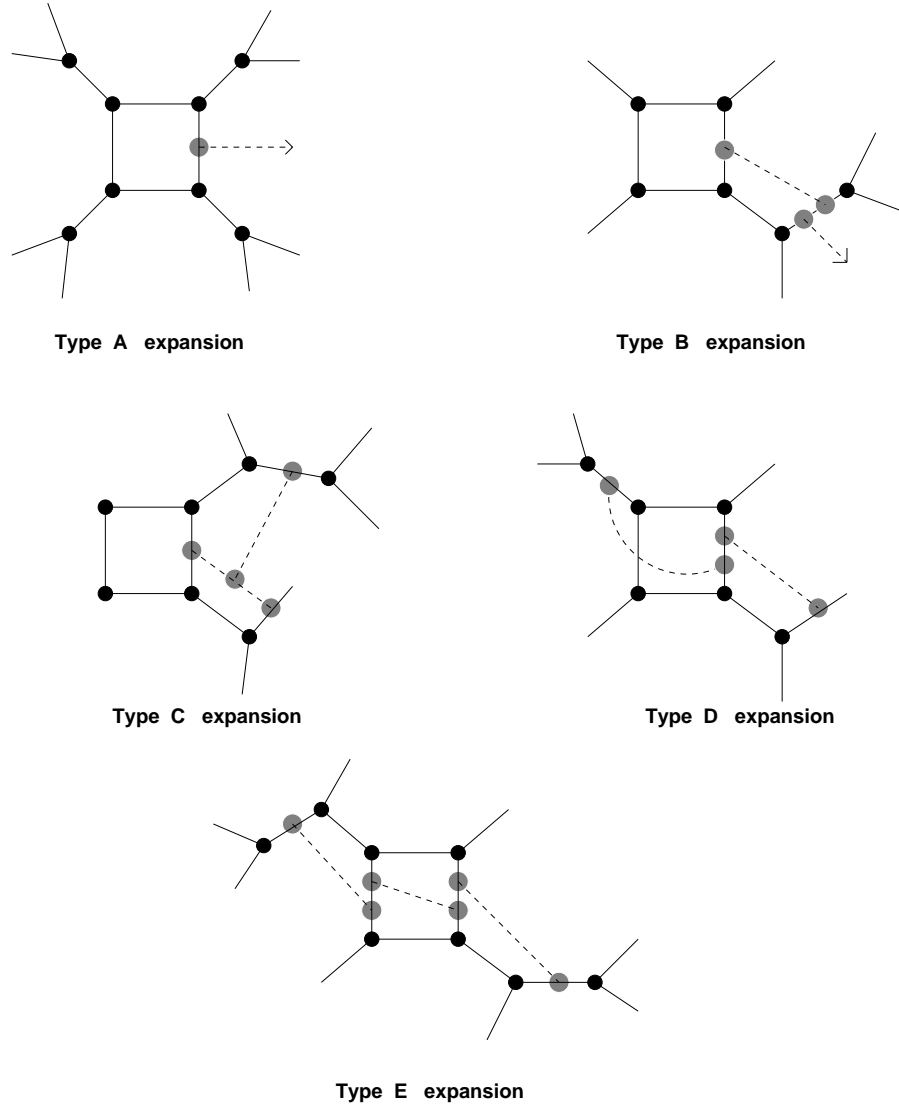


Figure 3

- Let  $G_3$  be  $G_1 + (k, l, v_2, v'_2)$  or  $G_1 + (u_1, v_1, v_4, v'_4)$ . If  $G_3$  is a long extension of  $G$  we say that  $G_3$  is a *type C expansion* of  $G$  based at  $C$ , and that the sequence  $G_1, G_3$  is a *standard generating sequence* of  $G_3$ .

(We apologize for the double use of the letter C and hope it causes no confusion.)

- Let  $G_4$  be the graph  $G_1 + (u_1, k, u_3, v_3)$ ; we say that  $G_4$  is a *type D expansion* of  $G$  based at  $C$ , and that the sequence  $G_1, G_4$  is a *standard generating sequence* of  $G_4$ .

- Let  $G'_5$  be the graph  $G_1 + (u_1, k, u_3, u_4)$ , let  $a, b$  be the new vertices of  $G'_5$ , and let  $G_5$  be the graph  $G'_5 + (b, u_4, v_3, v'_3)$ . The graph  $G_5$  is called a *type E expansion* of  $G$  based at  $C$ .

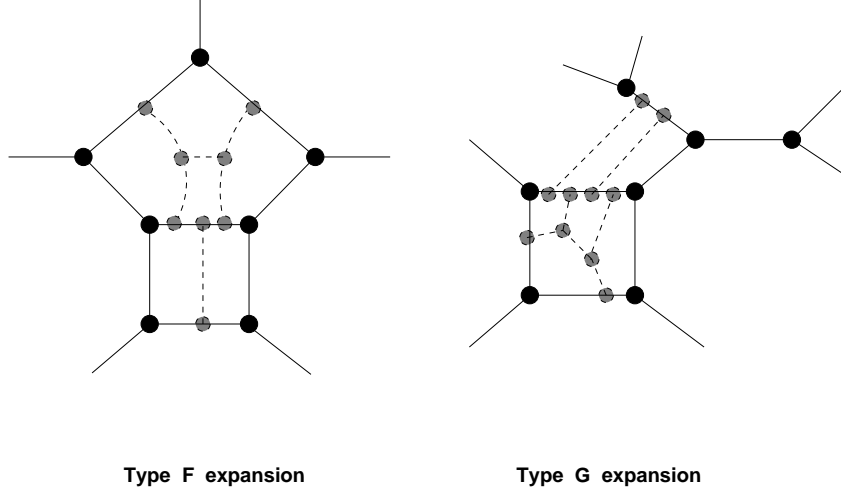


Figure 4

- If  $v'_1 = v'_2$ , then let  $G'_6 = G_1 + (u_2, k, v_2, v'_2)$ , let  $k_2, l_2$  be the new vertices of  $G'_6$ , let  $G''_6 = G'_6 + (k_2, k, u_3, u_4)$ , and let  $G_6 = G''_6 + (k, l, k_2, l_2)$ . The graph  $G_6$  is called a *type F expansion* of  $G$  based at  $C$ . We also say that  $G_6$  is a *type F expansion* of  $G$  based on  $(u_1, u_2)$ , and that  $k, l, k_2, l_2, k_3, l_3, k_4, l_4$  (in the order listed) are the *new vertices* of  $G_6$ , where  $k_3, l_3$  are the new vertices of  $G''_6$  and  $k_4, l_4$  are the new vertices of  $G_6$ . We say that

$$\{u_1, u_2, u_3, u_4, v_1, v_2, v'_1, k, l, k_2, l_2, k_3, l_3, k_4, l_4\}$$

is the *core* of the type F expansion  $G_6$ .

- Let  $G_7$  be a type F expansion of  $G_1$  based on  $(u_1, k)$ . We say that  $G_7$  is a *type G expansion* of  $G$  based at  $C$ .

(Again, apologies for the double use of the letter G.)

- Assume now that  $G$  has a quadrangle  $D$  with vertex-set  $x_1, x_2, x_3, x_4 \in V(G) - V(C)$  such that  $x_1$  is adjacent to  $u_1$ , the vertices  $u_2$  and  $x_2$  have a common neighbor, and

$u_4$  and  $x_4$  have a common neighbor. Assume further that  $\{x, y\} = \{x_1, x_2\}$ , and let us recall that  $k, l$  are the new vertices of  $G_1$ . Let  $G_8$  be a type F expansion of  $G_1$  based on  $(x_1, l)$ ; in those circumstances we say that  $G_8$  is a *type H expansion* of  $G$ . The quad-connectivity of  $G$  implies that in this case  $|V(G)| \leq 14$ .

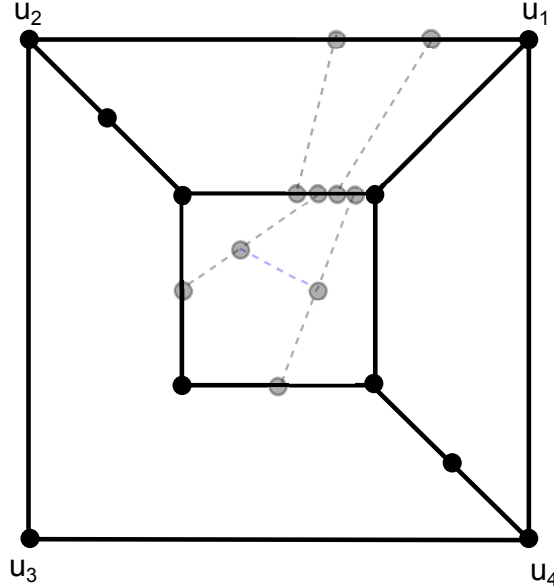


Figure 5. Type H expansion

It follows from the quad-connectivity of  $G$  that  $G_4, G_5, G_6, G_7, G_8$  are long extensions of  $G$ . We offer the following easy but important remark. Let us recall that generating sequences were defined at the beginning of Section 3.

**(4.2)** *Let  $G_0$  be a quad-connected cubic graph, let  $C_0$  be a quadrangle in  $G_0$ , let  $G'$  be a type A, B, C, D, E, F, G or H expansion of  $G_0$  based at  $C_0$ , let  $G_1, G_2, \dots, G_k$  be a generating sequence of  $G'$  from  $G$  based at  $C_0$ , and let  $F$  be a graph of minimum degree at least two. Let the vertices of  $C_0$  be  $u_1, u_2, u_3, u_4$  in order, let  $v_1$  be the neighbor of  $u_1$  not on  $C_0$ , and let  $v'_1, v''_1$  be the two neighbors of  $v_1$  other than  $u_1$ . If  $G_1 = G_0 + (u_1, u_2, v_1, v'_1)$ , then there exists a generating sequence  $G'_1, G'_2, \dots, G'_k$  of  $G'$  from  $G_0$  based at  $C_0$  such that*

- *for  $i = 1, 2, \dots, k$  the graph  $G'_i$  is isomorphic to  $G_i$ ,*
- $G'_1 = G'_0 + (u_1, u_4, v_1, v''_1)$ ,

- if  $F$  is a subgraph of both  $G_0$  and  $G_k$ , then  $F$  is a subgraph of  $G'_k$ , and
- if for some  $i \in \{1, 2, \dots, k\}$  the sequence  $G_i, G_{i+1}, \dots, G_k$  is a standard generating sequence of  $G_k$ , then the sequence  $G'_i, G'_{i+1}, \dots, G'_k$  is a standard generating sequence of  $G'_k$ .

The proof is clear.

**(4.3)** Let  $F$  be a graph of minimum degree at least two, let  $G$  be a quad-connected cubic graph, let  $C$  be a quadrangle in  $G$ , and let  $G_2$  be a long 2-extension of  $G$  based at  $C$  such that  $F$  is a subgraph of both  $G$  and  $G_2$  and  $F$  is disjoint from  $C$ . Then there exist an expansion  $G'$  of  $G$  of type A, B, C, or D based at  $C$ , and a homeomorphic embedding  $\eta' : G' \hookrightarrow G_2$  such that  $\eta'$  fixes  $F$ .

*Proof.* Let  $G, C, G_2$  be as stated, let  $u_1, u_2, u_3, u_4$  be the vertices of  $C$  (in order), and let  $v_i$  be the neighbor of  $u_i$  not on  $C$ . Let  $G_2 = G_1 + (u, v, x, y)$  and  $G_1 = G + (u_1, u_4, v_1, v'_1)$ , where  $v'_1 \notin V(C)$  is adjacent to  $v_1$ , and  $\{u, v\}$  is one of  $\{v_1, l_1\}, \{k_1, l_1\}, \{u_1, v_1\}, \{u_1, k_1\}$ , where  $k_1, l_1$  are the new vertices of  $G_1$ . Let  $k_2, l_2$  be the new vertices of  $G_2$ .

First, if  $\{u, v\} = \{v_1, l_1\}$ , then  $G_2$  is a type B expansion of  $G$ , and hence  $G_2$  and the identity homeomorphic embedding  $G_2 \hookrightarrow G_2$  satisfy the conclusion of the lemma. Second, let us assume that  $\{u, v\} = \{k_1, l_1\}$ . By considering the path  $k_1 k_2 l_2$  we see that  $\eta$  extends to a homeomorphic embedding  $G + (u_1, u_4, x, y) \hookrightarrow G_2$  that fixes  $F$ , and hence we may assume that the 1-extension  $G + (u_1, u_4, x, y)$  is short. It follows that  $\{x, y\}$  equals one of  $\{v_4, v'_4\}, \{u_3, v_3\}, \{u_2, u_3\}$  or  $\{u_2, v_2\}$ , where  $v'_4 \neq u_4$  is a neighbor of  $v_4$ . We break the analysis into three subcases. First, if  $\{x, y\} = \{v_4, v'_4\}$ , then  $G_2$  is a type C expansion of  $G$ , and hence  $G_2$  and the identity homeomorphic embedding  $G_2 \hookrightarrow G_2$  satisfy the conclusion of the lemma. For the second subcase assume that  $\{x, y\} = \{u_3, v_3\}$  or  $\{u_2, u_3\}$ . Let  $G' = G + (u_3, u_4, v_1, v'_1)$ ; then  $G'$  is a long 1-extension of  $G$  by (2.6), and hence is a type A expansion of  $G$ . Let  $\eta : G \hookrightarrow G_2$  be the canonical homeomorphic embedding determined by the generating sequence  $G_1, G_2$ . Let  $\eta' : G' \hookrightarrow G_2$  be obtained from  $\eta$  first by rerouting  $u_3 u_4$  along  $k_1 k_2 l_2$ , and then routing the new edge along  $k_2 l_1$ . Since  $u_1, u_2, u_3, u_4, k_1, l_1, k_2, l_2 \notin F$  we deduce that  $\eta'$  fixes  $F$ . (In the future we will omit this kind of argument, because it will be clear that all the homeomorphic embeddings that we

will construct will fix  $F$ .) The pair  $G', \eta'$  satisfies the conclusion of the lemma. The third and last subcase is that  $\{x, y\} = \{u_2, v_2\}$ . Let  $G' = G + (u_2, u_3, v_1, v_1'')$ , where  $v_1'' \notin \{v_1', u_1\}$  is the third neighbor of  $v_1$ . Then  $G'$  is a long 1-extension of  $G$  by (2.6). Let  $\eta' : G' \hookrightarrow G_2$  be obtained from  $\eta$  first by rerouting  $\eta(u_1 v_1)$  along  $k_1 k_2 l_1$ , then rerouting  $\eta(k_1 u_1 u_2)$  along  $k_2 l_2$ , and finally routing the new edge along  $\eta(u_2 u_1 v_1)$ . Then  $G', \eta'$  satisfy the conclusion of the lemma. This completes the case  $\{u, v\} = \{k_1, l_1\}$ .

The third case  $\{u, v\} = \{u_1, v_1\}$  is symmetric to the previous case by (4.2), and so we proceed to the fourth and last case, namely  $\{u, v\} = \{u_1, k_1\}$ . Let  $G'$  and  $\eta' : G' \hookrightarrow G_2$  be obtained from  $G, \eta$  by routing the new edge along  $\eta(k_2 l_2)$ . We may assume that  $G'$  is a short 1-extension of  $G$ , for otherwise the lemma holds. Thus either  $\{x, y\} = \{u_3, v_3\}$ , or  $\{x, y\} = \{v_4, v_4'\}$ , where  $v_4' \neq u_4$  is a neighbor of  $v_4$ . In the former case  $G_2$  is a type D expansion of  $G$ , and so the lemma holds, and hence we may assume that the latter case holds. Since by (4.2) there is symmetry between  $u_1 v_1$  and  $k_1 l_1$  we deduce that also  $\{x, y\} = \{v_2, v_2'\}$ , where  $v_2' \neq u_2$  is a neighbor of  $v_2$ . It follows that  $v_2$  and  $v_4$  are adjacent in  $G$ , contrary to the quad-connectivity of  $G$ . This completes the fourth case, and hence the proof of the lemma.  $\square$

**(4.4)** *Let  $F$  be a graph of minimum degree at least two, let  $G$  be a quad-connected cubic graph, let  $C$  be a quadrangle in  $G$ , and let  $G_3$  be a long 3-extension of  $G$  based at  $C$  such that  $F$  is a subgraph of both  $G$  and  $G_3$  and  $F$  is disjoint from  $C$ . Then there exist a graph  $G'$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow G_3$  such that  $\eta'$  fixes  $F$  and  $G'$  is either a type E expansion or a long 1- or 2-extension of  $G$  based at  $C$ .*

*Proof.* Let  $G_1$  be a short 1-extension of  $G$  based at  $C$  such that  $G_3$  is a long 2-extension of  $G_1$  based at the new quadrangle  $C_1$  of  $G_1$ . By (4.3) applied to the graph  $G_1$  and circuit  $C_1$  there exist an expansion  $G'_3$  of  $G_1$  of type A, B, C, or D based at  $C_1$ , and a homeomorphic embedding  $\eta'_3 : G'_3 \hookrightarrow G_3$  that fixes  $F$ . If  $G'_3$  is of type A, then it is a long 2-extension of  $G$  based at  $C$ , and the lemma holds. Thus we may assume that  $G'_3$  is of type B, C, or D. It follows that  $G_3$  and  $G'_3$  have the same number of vertices, and hence  $\eta'_3$  is an isomorphism of  $G_3$  and  $G'_3$ . It follows that if the conclusion of the lemma holds for  $G'_3$ , then it holds for  $G_3$ . Therefore we may assume that  $G'_3 = G_3$  and that  $\eta'_3$  is the identity homeomorphic

embedding. In other words,  $G_3$  is a type B, C, or D expansion of  $G_1$  based at  $C_1$ .

Let  $G_2, G_3$  be a standard generating sequence of the expansion  $G_3$ . Then  $G_1, G_2, G_3$  are quad-connected by (2.1). Let the vertices of  $C$  be  $u_1, u_2, u_3, u_4$  in order. For  $i = 1, 2, 3, 4$  let  $v_i$  be the neighbor of  $u_i$  not on  $C$ , let  $v'_i, v''_i$  be the neighbors of  $v_i$  different from  $u_i$ , and let  $w_i, w'_i$  be the neighbors of  $v''_i$  different from  $v_i$ . Let  $G_2 = G_1 + (a_1, a_2, a_3, a_4)$ . We may assume that  $G_1 = G + (u_1, u_4, v_1, v'_1)$ . Let  $u_5, v_5$  be the new vertices of  $G_1$ ; then  $V(C_1) = \{v_1, u_1, u_5, v_5\}$ . We claim the following.

- (1) *We may assume that  $\{a_1, a_2\} = \{u_1, v_1\}$ , and that  $\{a_3, a_4\}$  is equal to one of  $\{u_2, u_3\}$ ,  $\{u_2, v_2\}$  or  $\{v''_1, w'_1\}$ .*

To prove (1) we first note that by (4.2) there is symmetry between  $\{u_5, v_5\}$  and  $\{u_1, v_1\}$ , and so we may assume that  $\{a_1, a_2\} \neq \{u_5, v_5\}$ . Secondly, assume that  $\{a_1, a_2\} = \{v_1, v_5\}$ . Since  $G_2$  is not a long extension of  $G_1$ , one of  $a_3, a_4$  equals one of  $v'_1, v''_1$ . Let us assume that  $a_3 = v''_1$ ; the argument for  $v'_1$  is symmetric. We may assume from the symmetry that  $a_4 = w'_1$ . It follows from (4.2) applied to the graph  $G_1$  and cycle  $C_1$  that we may replace  $G_2$  by the graph  $G_1 + (u_1, v_1, v''_1, w_1)$  and thus arrange for the first assertion of (1) to hold. The case  $\{a_1, a_2\} = \{u_1, u_5\}$  follows similarly. This proves that we may assume that  $\{a_1, a_2\} = \{u_1, v_1\}$ . Since  $G_2$  is a short extension of  $G_1$ , we see that  $\{a_3, a_4\}$  is equal to one of  $\{u_2, u_3\}$ ,  $\{u_2, v_2\}$ ,  $\{v''_1, w'_1\}$  or  $\{v''_1, w_1\}$ . Since the last two cases are symmetric, we may assume that one of the first three occurs. This proves (1).

Let  $k_2, l_2$  be the new vertices of  $G_2$ , let  $G_3 = G_2 + (a_5, a_6, x, y)$ , and let  $k_3, l_3$  be the new vertices of  $G_3$ . Let  $\eta : G \hookrightarrow G_3$  be the canonical homeomorphic embedding determined by the generating sequence  $G_1, G_2, G_3$ . Since  $F$  has minimum degree at least two and is a subgraph of both  $G$  and  $G_3$  we deduce that

- (2)  $u_1, u_2, u_3, u_4, u_5, v_1, v_5, k_2, l_2, k_3, l_3 \notin F$ .

To make the forthcoming case analysis easier to follow let us make an outline. There will be three supercases depending on  $\{a_3, a_4\}$ . These will be divided into cases depending on the type of the expansion  $G_3$ , and the cases will sometimes be further divided into subcases depending on  $G_3$ . In each subcase we shall construct a pair  $G', \eta'$  that satisfies the conclusion of the theorem. We first dispose of the supercase  $\{a_3, a_4\} = \{u_2, u_3\}$ . Let

$G' = G + (u_2, u_3, v_1, v_1'')$ , and let  $\eta' : G' \hookrightarrow H$  be obtained from  $\eta$  by rerouting  $u_1v_1$  along  $u_5v_5$  and then routing the new edge along the path  $l_2k_2v_1$ . By (2.6) and (2) the pair  $G', \eta'$  satisfies the conclusion of the lemma. This completes the first supercase.

For the second supercase we assume that  $\{a_3, a_4\} = \{u_2, v_2\}$ . This will be divided into cases. As a first case assume that  $G_3$  is a type B expansion of  $G_1$ . Then  $\{a_5, a_6\} = \{l_2, u_2\}$ . Assume as a first subcase that  $\{x, y\}$  is not equal to any of  $\{v_3, v_3'\}$ ,  $\{v_3, v_3''\}$ ,  $\{u_4, v_4\}$ , or  $\{u_4, u_5\}$ . Let  $G', \eta'$  be obtained from  $\eta$  first by rerouting  $u_1u_2$  along  $l_2k_2$  and then by routing the new edge along  $k_3l_3$  if  $\{x, y\} \neq \{u_5, v_5\}$  and along  $k_3l_3v_5$  otherwise. Then  $G'$  is a long 1-extension of  $G$ , and so the lemma holds. We may therefore assume that  $\{x, y\}$  is equal to one of the sets specified above. As a second subcase assume that  $\{x, y\} = \{v_3, v_3'\}$  or  $\{x, y\} = \{v_3, v_3''\}$ . Then  $G_3$  is isomorphic to a type E expansion of  $G$ , and so the lemma holds. Thirdly, let us assume that  $\{x, y\} = \{u_4, v_4\}$  or  $\{x, y\} = \{u_4, u_5\}$ . Let  $G' = G_1 + (u_1, u_5, u_3, v_3)$ ; then  $G'$  is a long 2-extension of  $G$ . Thus  $G'$  and the homeomorphic embedding obtained from the canonical homeomorphic embedding  $G_1 \hookrightarrow G_3$  (determined by the generating sequence  $G_2, G_3$ ) first by rerouting  $u_1u_2$  along  $l_2k_2$ , then rerouting  $u_3u_4$  along  $k_3l_3$ , and finally routing the new edge along  $u_1u_2$  satisfy the conclusion of the lemma. This completes the case when  $G_3$  is a type B expansion.

For the second case assume that  $G_3$  is a type C expansion of  $G_1$ . There are two subcases. Assume first that  $\{a_5, a_6\} = \{u_1, u_2\}$ . Then  $\{x, y\} = \{u_4, v_4\}$ , because  $\{x, y\} \neq \{u_3, u_4\}$ , since  $G_3$  is a long 1-extension. Let  $G' = G + (u_3, u_4, v_1, v_1')$  and let  $\eta' : G' \hookrightarrow H$  be obtained from  $\eta$  first by rerouting  $u_1u_4$  along  $k_3l_3$ , and then by routing the new edge along  $u_4u_5v_5$ . The graph  $G'$  is a long 1-extension of  $G$  by (2.6), a contradiction. The second subcase is that  $\{a_5, a_6\} = \{k_2, l_2\}$ . Then say  $x = v_1''$  and  $y$  is a neighbor of  $v_1''$  different from  $v_1$ . Since  $G_3$  is a long extension of  $G$  we deduce that  $v_2 \neq y$ . Let  $G' = G + (u_1, u_2, v_1'', y)$  and let  $\eta' : G' \hookrightarrow H$  be obtained from  $\eta$  first by rerouting  $u_1u_2$  along  $l_2k_2$  and then routing the new edge along  $k_3l_3$ . Then  $G'$  is a long 1-extension of  $G$  by (2.6), a contradiction. This completes the second case. For the third case we assume that  $G_3$  is a type D expansion of  $G_1$ . Then  $\{a_5, a_6\} = \{u_1, k_2\}$  and  $\{x, y\} = \{v_1', v_5\}$ . Let  $G' = G_1 + (u_4, v_4, v_1, v_5)$ ; then  $G'$  is a long 1-extension of  $G_1$  by (2.6), and hence it is a long 2-extension of  $G$ . Let  $\eta'$  be obtained from the canonical homeomorphic embedding  $G_1 \hookrightarrow G_3$  first by rerouting  $u_1u_2$



along  $k_2l_2$ , then rerouting  $u_5v_5$  along  $k_3l_3$ , then rerouting  $u_3u_4$  along  $u_1u_2$ , and finally routing the new edge along  $u_5v_5$ . Again, the pair  $G', \eta'$  satisfies the conclusion of the lemma. This completes the third case and hence the second supercase.

The third and last supercase is that  $\{a_3, a_4\} = \{v_1'', w_1'\}$ . We claim that we may assume that  $w_1' = v_2$ . Indeed, suppose that  $w_1' \neq v_2$ , let  $G' = G + (u_1, u_2, v_1'', w_1')$  and let  $\eta'$  be obtained from  $\eta$  by rerouting  $u_1v_1$  along  $u_5v_5$  and by routing the new edge along  $u_1k_2l_2$ . Then  $G', \eta'$  satisfy the conclusion of the lemma by (2.6). This proves that we may assume that  $w_1' = v_2$ . From the symmetry we may assume that  $v_2' = v_1''$ . We distinguish three cases depending on whether  $G_3$  is of type B, C, or D.

For the first case assume that  $G_3$  is of type B. Then  $\{a_5, a_6\} = \{l_2, v_1''\}$ . Let us first dispose of the case when one of  $x, y$  is equal to  $v_2''$ ; say  $x = v_2''$ . Then  $y \neq v_2$ , because  $G_3$  is a long extension of  $G$ . Let  $G'$  and  $\eta' : G' \hookrightarrow H$  be obtained from  $\eta$  by first rerouting  $v_2v_2''$  along  $k_3l_3$ , then rerouting  $u_1u_2$  along  $k_2l_2$ , then routing the first new edge along  $u_5v_5$ , then routing the second new edge along  $u_1u_2$ , and finally routing the third new edge along  $v_2v_2''$ . Then  $G'$  is a type E expansion of  $G$ , and thus  $G', \eta'$  satisfy the conclusion of the lemma. We may therefore assume that  $x, y \neq v_2'$ . Let us assume next that  $\{x, y\}$  is not equal to any of the pairs  $\{u_1, u_5\}, \{u_1, u_2\}, \{u_2, u_3\}$ . Let  $\xi$  be obtained from  $\eta$  by first rerouting  $\eta(u_1v_1)$  along  $u_5v_5$ , and then rerouting  $\eta(v_2v_2')$  along  $l_2k_2v_1$ . Let  $G''$  and  $\eta'' : G'' \hookrightarrow H$  be obtained from  $\xi$  by routing the first new edge along  $u_1k_2$ , and then routing the second new edge along  $l_2v_2'$ . Then  $G''$  is a long 2-extension of  $G$ , and hence the pair  $G'', \eta''$  satisfies the conclusion of the lemma. We may therefore assume that  $\{x, y\}$  is equal to one of the pairs  $\{u_1, u_5\}, \{u_1, u_2\}, \{u_2, u_3\}$ . Let  $G'''$  and  $\eta''' : G''' \hookrightarrow H$  be obtained from  $\xi$  by routing the new edge along  $v_2'k_3l_3$ . Then  $G'''$  is a long 1-extension of  $G$ , and hence the pair  $G''', \eta'''$  satisfies the conclusion of the lemma. This completes the first case.

For the second case assume that  $G_3$  is of type C. Then  $\{a_5, a_6\} = \{v_1, v_1''\}$  or  $\{a_5, a_6\} = \{k_2, l_2\}$ . Thus we distinguish two subcases. Assume as a first subcase that  $\{a_5, a_6\} = \{v_1, v_1''\}$ . It follows that one of  $x, y$  equals  $v_1'$ , say  $x = v_1'$ . Then  $y \neq v_5$ ; let  $y' \notin \{v_5, y\}$  be the third neighbor of  $v_1'$ . Let  $\zeta'$  be obtained from  $\eta$  by rerouting  $v_1v_1'$  along  $k_3l_3$ . Let  $G' = G + (u_1, u_4, v_1', y')$  and  $\eta' : G' \hookrightarrow H$  be obtained from  $\zeta'$  by routing the new edge along  $u_5v_5v_1'$ ; if  $v_4 \neq y'$  then  $G', \eta'$  satisfy the lemma by (2.6). Thus we may assume that

$y' = v_4$ . Let  $G'' = G + (u_1, u_4, v'_1, v_4) + (u_2, u_3, v_2, v'_2)$  and let  $\eta''$  be obtained from  $\zeta'$  first by rerouting  $u_1u_5$  along  $v_1v_5u_5$ , then by rerouting  $u_2v_2$  along  $k_2l_2$ , then routing the first new edge along  $v_5v'_1$ , and then routing the second new edge along  $u_2v_2$ . Let  $G', \eta'$  be obtained from  $\eta''$  by routing the new edge along  $\eta_3(u_1u_5)$ . Then  $G'$  is a type E expansion of  $G$ , and thus  $G', \eta'$  satisfy the conclusion of the lemma. This completes the first subcase. For the second subcase assume that  $\{a_5, a_6\} = \{k_2, l_2\}$ ; then  $\{x, y\} = \{u_2, u_3\}$ , because  $\{x, y\} \neq \{u_2, v_2\}$  by the fact that  $G_3$  is a long 1-extension of  $G$ . Let  $G' = G + (u_2, u_3, v_1, v'_1)$  and let  $\eta'$  be obtained from  $\eta$  by rerouting  $u_1v_1$  along  $u_5v_5$  and then routing the new edge along  $l_3k_3k_2v_1$ . Then  $G', \eta'$  satisfy the conclusion of the lemma by (2.6). This completes the second subcase and hence the second case.

For the third case assume that  $G_3$  is of type D. Then  $\{a_5, a_6\} = \{v_1, k_2\}$  and  $\{x, y\} = \{u_4, u_5\}$ . Let  $G' = G + (u_1, u_2, v_2, v'_2) + (a, u_2, v_1, v'_1)$  (where  $a, b$  are the new vertices of  $G + (u_1, u_2, v_2, v'_2)$ ) and  $\eta' : G' \hookrightarrow H$  be obtained from  $\eta$  by rerouting  $\eta(u_1u_4)$  along  $k_3l_3$ , then routing the first new edge along  $k_2l_2$ , and then routing the second new edge along  $u_1u_5v_5$ . Then  $G'$  is a long 2-extension of  $G$  by (2.6), and hence the pair  $G', \eta'$  satisfies the conclusion of the lemma. This completes the third case, and hence the third supercase, and thus the proof of the lemma.  $\square$

**(4.5)** *Let  $F$  be a graph of minimum degree at least two, let  $G$  be a quad-connected cubic graph, let  $C$  be a quadrangle in  $G$ , and let  $G_4$  be a long 4-extension of  $G$  based at  $C$  such that  $F$  is a subgraph of both  $G$  and  $G_4$  and  $F$  is disjoint from  $C$ . Then there exist a graph  $G'$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow G_4$  such that  $\eta'$  fixes  $F$  and  $G'$  is either a type F expansion of  $G$  based at  $C$  or a long  $n$ -extension of  $G$  based at  $C$  for some  $n \in \{1, 2, 3\}$ .*

*Proof.* Similarly as in the proof of (4.4) we may assume that there exists a short 1-extension  $G_1 = G + (u_1, u_4, v_1, v'_1)$  of  $G$  based at  $C$  such that  $G_4$  is a type E expansion of  $G_1$  based at the new quadrangle  $C_1$  of  $G_1$ . Since  $G_1$  is short one of  $u_1, u_4$  is adjacent to one of  $v_1, v'_1$ , and so we may assume that  $u_1$  is adjacent to  $v_1$ . Let the vertices of  $C$  be  $u_1, u_2, u_3, u_4$  in order, and for  $i = 1, 2, 3, 4$  let  $v_i$  be the neighbor of  $u_i$  not on  $C$ . Let  $u_5, v_5$  be the new vertices of  $G_1$ ; then  $V(C_1) = \{v_1, u_1, u_5, v_5\}$ . By (4.2) there is symmetry between  $u_1v_1$

and  $u_5v_5$ ; hence there are only two cases to consider, namely  $G_3 = G_2 + (u_1, v_1, u_2, u_3)$  and  $G_3 = G_2 + (u_1, v_1, u_2, v_2)$ , where  $G_2 = G_1 + (u_5, v_5, v'_1, w)$ ,  $G_4 = G_3 + (k_2, v_5, u_1, k_3)$ ,  $w$  is a neighbor of  $v'_1$  different from  $v_5$  and  $k_i, l_i$  are the new vertices of  $G_i$  for  $i = 1, 2, 3, 4$ . Let us first dispose of the former case. Let  $G', \eta'$  be obtained from  $\eta$  by first rerouting  $u_1v_1$  along  $u_5k_2k_4v_5$ , and then routing the new edge along  $l_3k_3v_1$ . Then by (2.6)  $G', \eta'$  satisfy the conclusion of the lemma and so we may assume that the latter case holds. We claim that we may assume that  $w = v_4$ . Otherwise let  $G' = G + (u_1, u_4, v'_1, w)$  and  $\eta'$  be obtained from  $\eta$  by routing the new edge along  $u_5k_2l_2$ ; then  $G', \eta'$  again satisfy the conclusion of the lemma by (2.6). Thus we may assume that  $w = v_4$ . Now  $G_4$  is isomorphic to a type F expansion of  $G$ , and so the conclusion of the lemma is satisfied.  $\square$

**(4.6)** *Let  $F$  be a graph of minimum degree at least two, let  $G$  be a quad-connected cubic graph, let  $C$  be a quadrangle in  $G$ , and let  $G_5$  be a long 5-extension of  $G$  based at  $C$  such that  $F$  is a subgraph of both  $G$  and  $G_5$  and  $F$  is disjoint from  $C$ . Then there exist a graph  $G'$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow G_5$  such that  $\eta'$  fixes  $F$  and  $G'$  is either a type G or type H expansion of  $G$  based at  $C$  or a long  $n$ -extension of  $G$  based at  $C$  for some  $n \in \{1, 2, 3, 4\}$ .*

*Proof.* Similarly as in the previous two proofs we may assume that there exists a short 1-extension  $G_1 = G + (u_1, u_4, v_1, v'_1)$  of  $G$  based at  $C$  such that  $G_5$  is a type F expansion of  $G_1$  based at the new quadrangle  $C_1$  of  $G_1$ . Since  $G_1$  is a short extension one of  $u_1, u_4$  is adjacent to one of  $v_1, v'_1$ , and so we may assume that  $u_1$  is adjacent to  $v_1$ . Let the vertices of  $C$  be  $u_1, u_2, u_3, u_4$  in order, and for  $i = 1, 2, 3, 4$  let  $v_i$  be the neighbor of  $u_i$  not on  $C$ . Let  $v''_1 \notin \{u_1, v'_1\}$  be the third neighbor of  $v_1$ . Let  $u_5, v_5$  be the new vertices of  $G_1$ ; thus  $V(C_1) = \{v_1, u_1, u_5, v_5\}$ . Since by (4.2) there is symmetry between  $u_1v_1$  and  $u_5v_5$  there are only three cases to consider, namely whether  $G_5$  is based on  $u_1u_5$ ,  $v_1v_5$  or  $u_5v_5$ . If  $G_5$  is based on  $u_1u_5$  then  $G_5$  is a type G expansion of  $G$ , and so  $G_5$  and the identity homeomorphic embedding satisfy the conclusion of the lemma.

Next we assume that  $G_5$  is based on  $v_1v_5$ . It follows that  $v'_1$  and  $v''_1$  have a common neighbor, say  $w$ , and  $\{v_1, v'_1, w, v''_1\}$  is the vertex-set of a quadrangle in  $G$ . Let  $z$  be the neighbor of  $v'_1$  in  $G$  other than  $v_1$  and  $w$ . By a rerouting argument similar to ones used in

previous proofs it is easy to construct a homeomorphic embedding  $G + (u_1, u_4, v'_1, z) \hookrightarrow G_5$  that fixes  $F$ . By (2.6) the lemma holds, unless  $z = v_4$ . Thus we may assume that  $z = v_4$ , and similarly that  $v_2$  and  $v'_1$  have a common neighbor. We deduce that  $G_5$  is a type H expansion of  $G$ , as desired.

We may therefore assume that  $G_5$  is based on  $u_5v_5$ . Then  $v_4$  and  $v'_1$  are adjacent. Let  $G'$  be obtained from  $G_5$  by deleting the edge  $u_2u_3$  and suppressing the resulting degree two vertices. Then  $G'$  is isomorphic to a type F expansion of  $G$ , and it is easy to construct a homeomorphic embedding  $\eta' : G' \hookrightarrow G_5$  that fixes  $F$ . Hence  $G', \eta'$  satisfy the conclusion of the lemma.  $\square$

**(4.7)** *Let  $F$  be a graph of minimum degree at least two, let  $G$  be a quad-connected cubic graph, let  $C$  be a quadrangle in  $G$ , let  $n \geq 1$  be an integer, and let  $G_3$  be a long  $n$ -extension of  $G$  based at  $C$  such that  $F$  is a subgraph of both  $G$  and  $G_3$  and  $F$  is disjoint from  $C$ . Then there exist a graph  $G'$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow G_3$  such that  $\eta'$  fixes  $F$  and  $G'$  is a long  $n'$ -extension of  $G$  based at  $C$  for some  $n' \in \{1, 2, 3, 4, 5\}$ .*

*Proof.* We proceed by induction on  $n$ . Similarly as in the previous three proofs we may assume that there exists a short 1-extension  $G_1 = G + (u_1, u_4, v_1, v'_1)$  of  $G$  based at  $C$  such that  $G_3$  is a type G or H expansion of  $G_1$  based at the new quadrangle  $C_1$  of  $G_1$ . Thus one of  $u_1, u_4$  is adjacent to one of  $v_1, v'_1$ , and so we may assume that  $u_1$  is adjacent to  $v_1$ . Let the vertices of  $C$  be  $u_1, u_2, u_3, u_4$  in order, and for  $i = 1, 2, 3, 4$  let  $v_i$  be the neighbor of  $u_i$  not on  $C$ . Let  $v''_1 \notin \{u_1, v'_1\}$  be the third neighbor of  $v_1$  in  $G$ . Let  $u_5, v_5$  be the new vertices of  $G_1$ . Thus  $V(C_1) = \{v_1, u_1, u_5, v_5\}$ . Then  $G_1$  is quad-connected by (2.1).

We first assume that  $G_3$  is a type G expansion of  $G_1$ . Let  $G_2$  be a 1-extension of  $G_1$  such that  $G_3$  is a type F expansion of  $G_2$ , and let  $k_2, l_2$  be the new vertices of  $G_2$ . From the symmetry it suffices to consider three subcases. We consider them separately in the next three paragraphs.

As a first subcase assume that  $G_2 = G_1 + (u_1, v_1, u_2, u_3)$ . Let  $G', \eta'$  be obtained from  $\eta$  by first rerouting  $\eta(u_1v_1)$  along  $\eta(u_5v_5)$ , and then routing the new edge along  $l_2k_2v_1$ . Then  $G', \eta'$  satisfy the conclusion of the lemma.

In the second subcase  $G_2 = G_1 + (u_1, v_1, u_2, v_2)$ , and  $G_3$  is based on  $(u_1, k_2)$ . Let  $G'$

be obtained from  $G_3$  by deleting the edge  $u_3u_4$ , and suppressing the resulting vertices of degree two. Then  $G'$  is isomorphic to a type G expansion of  $G$ , and it is easy to construct a homeomorphic embedding  $\eta' : G' \hookrightarrow G_3$  that fixes  $F$ . Then the pair  $G', \eta'$  satisfies the conclusion of the lemma.

In the third and last subcase  $G_2 = G_1 + (u_5, v_5, v'_1, z)$ , where  $z$  is a neighbor of  $v'_1$  different from  $v_5$ , and  $G_3$  is based on  $(v_5, k_2)$ . Let  $G' = G + (u_1, u_4, v'_1, z)$ ; by considering the path  $u_5k_2l_2$  it is easy to construct a homeomorphic embedding  $\eta' : G' \hookrightarrow G_3$  that fixes  $F$ . If  $z \neq v_4$  then by (2.6) the pair  $G', \eta'$  satisfies the conclusion of the lemma. We may therefore assume that  $z = v_4$ . Let  $G'$  be obtained from  $G_3$  by deleting the edges  $u_4v_4$  and  $u_2u_3$  and suppressing the resulting degree two vertices. and let  $\eta'$  be the canonical homeomorphic embedding  $G' \hookrightarrow G_3$ . Then  $G'$  is isomorphic to a type F expansion of  $G$ , and it is easy to construct a homeomorphic embedding  $\eta' : G' \hookrightarrow G_3$  that fixes  $F$ . Hence the pair  $G', \eta'$  satisfies the conclusion of the lemma.

We now assume that  $G_3$  is a type H expansion of  $G_1$  based at  $C_1$ . By (4.2) there is symmetry between  $u_1$  and  $u_5$ . Let  $D$  be as in the definition of expansion of type H. Thus some vertex of  $D$  is adjacent in  $G_1$  to some vertex of  $C$ . By symmetry it suffices to consider only two subcases. In the first subcase  $v''_1 \in V(D)$  is adjacent to  $v_1 \in V(C_1)$ , and a neighbor of  $v''_1$  in  $D$  is adjacent to  $v'_1$ . But then the set  $V(D) \cup \{v_1, v'_1\}$  contradicts the quad-connectivity of  $G$ . In the second subcase some vertex of  $D$  is adjacent to  $u_5$ , and the set  $V(D) \cup \{u_1, u_2\}$  contradicts the quad-connectivity of  $G$ .  $\square$

**(4.8)** *Let  $F$  be a graph of minimum degree at least two, let  $G$  be a quad-connected cubic graph, let  $C$  be a quadrangle in  $G$ , let  $n \geq 1$  be an integer, and let  $G_3$  be a long  $n$ -extension of  $G$  based at  $C$  such that  $F$  is a subgraph of both  $G$  and  $H$  and  $F$  is disjoint from  $C$ . Then there exist a graph  $G'$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow G_3$  such that  $\eta'$  fixes  $F$  and  $G'$  is a type A, B, C, D, E, F, G or H expansion of  $G$  based at  $C$ .*

*Proof.* Let us choose an integer  $n_2 \geq 1$ , a graph  $G_2$  and a homeomorphic embedding  $\eta_2 : G_2 \hookrightarrow G_3$  such that  $G_2$  is a long  $n_2$ -extension of  $G$  based at  $C$ , the homeomorphic embedding  $\eta_2$  fixes  $F$ , and, subject to that,  $n_2$  is minimum. Such a choice is possible, because  $n_2 = n$ ,  $G_2 = G_3$  and the identity homeomorphic embedding satisfy the requirements

(except minimality).

We claim that there do not exist an integer  $n_1$ , graph  $G_1$  and homeomorphic embedding  $\eta_1 : G_1 \hookrightarrow G_2$  such that  $1 \leq n_1 < n_2$ ,  $G_1$  is a long  $n_1$ -extension of  $G$  based at  $C$  and  $\eta_1$  fixes  $F$ . Indeed, otherwise the graph  $G_1$  and homeomorphic embedding  $\eta_1 \circ \eta_2$  violate the choice of  $G_2, \eta_2$ . This proves our claim that  $n_1, G_1, \eta_1$  do not exist.

It follows from (4.3), (4.4), (4.5), (4.6), and (4.7) that there exist a graph  $G'$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow G_2$  such that  $\eta'$  fixes  $F$  and  $G'$  is a type A, B, C, D, E, F, G or H expansion of  $G$  based at  $C$ . Thus  $G'$  and the homeomorphic embedding  $\eta' \circ \eta_2$  satisfy the conclusion of the lemma.  $\square$

**(4.9)** *Let  $G, H$  be cubic graphs, let  $F$  be a graph of minimum degree at least two, let  $\eta : G \hookrightarrow H$  fix  $F$ , let  $C$  be a quadrangle in  $G$  disjoint from  $F$ , let  $G$  be quad-connected, and let  $H$  be cyclically 5-connected. Then there exist an expansion  $G'$  of  $G$  based at  $C$  of type A, B, C, D, E, F, G, or H and a homeomorphic embedding  $\eta' : G' \hookrightarrow H$  that fixes  $F$ .*

*Proof.* By (4.1) there exist an integer  $n \geq 1$ , a long  $n$ -extension  $G_2$  of  $G$  based at  $C$ , and a homeomorphic embedding  $\eta_2 : G_2 \hookrightarrow H$  that fixes  $F$ . By (4.8) there exist a type A, B, C, D, E, F, G, or H expansion  $G_1$  of  $G$  based at  $C$  and a homeomorphic embedding  $\eta_1 : G_1 \hookrightarrow G_2$  that fixes  $F$ . Thus  $G_1$  and  $\eta_1 \circ \eta_2$  are as desired.  $\square$

When  $F$  is the null graph we obtain the following corollary.

**(4.10)** *Let  $G, H$  be cubic graphs, let  $G \hookrightarrow H$ , let  $C$  be a quadrangle in  $G$ , let  $G$  be quad-connected, and let  $H$  be cyclically 5-connected. Then there exist an expansion  $G'$  of  $G$  based at  $C$  of type A, B, C, D, E, F, G, or H and a homeomorphic embedding  $G' \hookrightarrow H$ .*

## 5. DODECAHEDRAL CONNECTION

In this section we introduce dodecahedral connectivity, a notion of connectivity that is stronger than cyclic 5-connectivity. The main result of this section, (5.6) below, says that if the graph  $H$  in (4.9) is dodecahedrally connected, then the last three outcomes of (4.9) can be eliminated.

A *guild* is a pair  $(G, \pi)$ , where  $G$  is a graph with every vertex of degree 1 or 3, and  $\pi$  is a cyclic ordering of the set of vertices of  $G$  of degree 1. (We consider  $(1, 2, 3, 4, 5)$  and  $(3, 2, 1, 5, 4)$  to be the same cyclic ordering.) This is closely related to the notion of a society, introduced in [8]. If  $(G, \pi)$  and  $(G', \pi')$  are guilds and  $\eta : G \hookrightarrow G'$  is a homeomorphic embedding, we say that  $\eta$  is a *homeomorphic embedding of  $(G, \pi)$  into  $(G', \pi')$*  if  $\eta$  takes  $\pi$  onto  $\pi'$ . If that is the case we write  $\eta : (G, \pi) \hookrightarrow (G', \pi')$ . If  $\delta A$  is an edge-cut of a cubic graph  $G$  of cardinality  $k$  such that  $\delta A$  is a matching, and  $v_1, v_2, \dots, v_k$  are all the vertices of  $V(G) - A$  incident with an edge of  $\delta A$ , then let  $H$  be the graph  $G|(A \cup \{v_1, v_2, \dots, v_k\})$ . We say that  $(H, (v_1, v_2, \dots, v_k))$  is a *shore guild corresponding to  $A$* . Thus if  $k > 2$  there are  $(k - 1)!/2$  shore guilds corresponding to  $A$ .

Let  $G$  be Dodecahedron, and let  $C$  be a circuit of  $G$  of length five with vertices  $u_1, u_2, \dots, u_5$  in order. For  $i = 1, 2, \dots, 5$  let  $v_i$  be the neighbor of  $u_i$  not on  $C$ . Let  $G'$  be the graph obtained from  $G$  by deleting the edges of  $C$ ; then  $D = (G', (u_1, u_2, u_3, u_4, u_5))$  is a guild, called the *Dodecahedron guild*. Let  $G'' = G' + (u, v, x, y)$  be a 1-extension of  $G'$ . We say that  $D' = (G'', (u_1, u_2, u_3, u_4, u_5))$  is a *non-planar expansion of the Dodecahedron guild* if  $\{u, v\} \neq \{u_i, v_i\}$  for all  $i = 1, 2, \dots, 5$ , and neither  $u$  nor  $v$  is equal or adjacent to  $x$  or  $y$ .

Let  $G$  be a cyclically 5-connected cubic graph. We say that  $G$  is *dodecahedrally connected* if for every edge-cut  $\delta A$  of cardinality five and every shore guild  $S$  corresponding to  $A$ , if  $\eta : D \hookrightarrow S$  is a homeomorphic embedding of the Dodecahedron guild into  $S$ , then there exist a non-planar expansion  $D'$  of  $D$  and a homeomorphic embedding  $\eta' : D' \hookrightarrow S$ .

The following proposition from [10] is not needed in this paper, but is stated for the reader's convenience as it sheds some light on the seemingly mysterious definition of dodecahedral connection. A guild  $(G, \pi)$  is *planar* if  $G$  can be drawn in a closed disc  $\Delta$  with the vertices of degree one drawn in the boundary of  $\Delta$  in the order given by  $\pi$ .

**(5.1)** *A cyclically 5-connected cubic graph  $G$  is dodecahedrally connected if and only if for every edge-cut  $\delta A$  of cardinality 5 with  $|A| \geq 7$  and  $|V(G) - A| \geq 7$ , no shore guild corresponding to  $A$  is planar.*

We need the following lemma.

**(5.2)** Let  $G, G_1, H$  be cubic graphs, let  $G$  be quad-connected, let  $F$  be a graph of minimum degree at least two, let  $C$  be a quadrangle in  $G$ , let  $G_1$  be a type  $F$  expansion of  $G$  with core  $R$  based at  $C$  such that  $R$  is disjoint from  $F$ , let  $H$  be dodecahedrally connected, and let  $\eta_1 : G_1 \hookrightarrow H$  be a homeomorphic embedding that fixes  $F$ . Then there exist a 1-extension  $G_2 = G_1 + (u, v, x, y)$  of  $G_1$  and a homeomorphic embedding  $G_2 \hookrightarrow H$  that fixes  $F$  and such that  $u, v \in R$ , and either  $G_2$  is a long 1-extension of  $G_1$  or  $x, y \notin R$ .

*Proof.* Let  $G, G_1, H, C, R, \eta_1$  be as stated. Then  $\delta R$  is an edge-cut of  $G_1$  of cardinality five such that some shore guild corresponding to  $R$  is isomorphic to the Dodecahedron guild. Let  $\delta R = \{e_1, e_2, \dots, e_5\}$ . If there exists an edge-cut  $\delta A$  of  $H$  of cardinality five with  $\eta_1(R) \subseteq A$  and  $\eta_1(V(G_1) - R) \subseteq V(H) - A$  then the conclusion follows from the definition of dodecahedral connection. We may therefore assume that no such edge-cut exists. Thus by (3.1) there exists an augmenting sequence  $\gamma = (Q_1, Q_2, \dots, Q_n)$  with respect to  $G_1, H, R$  and  $\eta_1$ . By (3.2) we may assume (by replacing  $\eta_1$  by a different embedding if necessary) that the conclusion of (3.2) holds. Let  $G_2, \eta_2$  be obtained from  $\eta_1$  by routing the new edge along  $Q_1$ ; it follows that  $G_2$  and  $\eta_2$  satisfy the conclusion of the lemma.  $\square$

The following result will allow us to eliminate type  $F$  expansions when the graph  $H$  is dodecahedrally connected.

**(5.3)** Let  $G, G_4, H$  be cubic graphs, let  $C$  be a quadrangle in  $G$ , let  $F$  be a graph of minimum degree at least two, let  $G$  be quad-connected, let  $G_4$  be a type  $F$  expansion of  $G$  based at  $C$  with core  $R$ , and let  $G_5 = G_4 + (u, v, x, y)$  be a 1-extension of  $G_4$  such that  $u, v \in R$ , and either  $G_5$  is a long 1-extension of  $G_4$  or  $x, y \notin R$ . Assume further that  $F$  is a subgraph of both  $G$  and  $G_5$ . Then there exist an integer  $n \in \{1, 2, 3\}$ , a long  $n$ -extension  $G'$  of  $G$  based at  $C$ , and a homeomorphic embedding  $\eta' : G' \hookrightarrow G_5$  that fixes  $F$ .

*Proof.* Let  $u_1, u_2, u_3, u_4$  be the vertices of  $C$  in order, for  $i = 1, 2, 3, 4$  let  $v_i$  be the neighbor of  $u_i$  not on  $C$ , and let  $v'_i, v''_i$  be the neighbors of  $v_i$  other than  $u_i$ . Since  $G$  has a type  $F$  expansion we may assume that  $v'_1 = v'_2$ . Let  $w \notin \{v_1, v_2\}$  be the third neighbor of  $v'_1$ . Choose  $G_1, G_2, G_3$  such that each of  $G, G_1, G_2, G_3, G_4, G_5$  is a 1-extension of the previous. For  $i = 0, 1, 2, 3, 4$  let  $\eta_i$  be the canonical homeomorphic embedding  $G_i \hookrightarrow G_5$  determined by the generating sequence  $G_{i+1}, G_{i+2}, \dots, G_5$ , where  $G_0$  means  $G$ ,



for  $i = 1, 2, 3, 4$  let  $k_i, l_i$  be the new vertices of  $G_i$ , and let  $G_1 = G + (u_1, u_2, v_1, v'_1)$ ,  $G_2 = G_1 + (k_1, u_2, v_2, v'_1)$ ,  $G_3 = G_2 + (k_1, k_2, u_3, u_4)$  and  $G_4 = G_3 + (k_1, l_1, k_2, l_2)$ . Then  $R = \{u_1, u_2, u_3, u_4, v_1, v_2, v'_1, k_1, l_1, k_2, l_2, k_3, k_4, l_4\}$ . From (2.6) we deduce that

- (1) *the vertices in  $R \cup \{v''_1, w, v''_2, v_3, v_4\}$  are pairwise distinct, except that possibly  $w = v_3$  or  $w = v_4$ , but not both.*

We also point out for future reference that

- (2) *there is symmetry fixing  $v'_1, w$  and taking  $u_1, u_4, v_1, v''_1, v_4$  onto  $u_2, u_3, v_2, v''_2, v_3$ , respectively.*
- (3) *If  $u \in \{k_1, k_2, k_3, k_4, l_4\}$  and  $x, y \notin R - \{v'_1\}$  then the lemma holds.*

To prove (3) let  $u, x, y$  be as stated, and let  $G' = G + (u_1, u_2, x, y)$  and  $\eta'$  be obtained from  $\eta_0$  by routing the new edge along  $\eta_5(k_5l_5) \cup Q$ , where  $Q$  is an appropriate subpath of  $\eta_4(G_4)$ . Then  $G', \eta'$  satisfy the conclusion of the lemma, and (3) follows.

- (4) *If  $u = l_3$ ,  $\{x, y\} \cap \{v_3, v_4\} = \emptyset$ ,  $\{x, y\} \neq \{u_1, k_1\}$  and  $\{x, y\} \neq \{u_2, k_2\}$  then the lemma holds.*

To prove (4) we may assume by (2) that  $\{x, y\}$  does not equal  $\{u_2, v_2\}$ ,  $\{k_1, k_4\}$  or  $\{k_4, l_1\}$ . Let  $G', \eta'$  be obtained from  $\eta_0$  by rerouting  $\eta_5(u_1v_1)$  along  $\eta_1(k_1l_1)$ , and then by routing the new edge along  $Q \cup \eta_5(k_5l_5) \cup Q'$ , where  $Q$  is  $\eta_5(k_5l_3)$  if  $v = k_3$  and null otherwise, and  $Q'$  is  $\eta_5(l_5v_1)$  if  $\{x, y\} = \{u_1, v_1\}$ , a subpath of  $\eta_4(k_2l_4) \cup \eta_4(l_4l_2) \cup \eta_4(l_4k_4)$  with ends  $\eta_5(l_5)$  and  $\eta_4(l_2)$  if  $l_4 \in \{x, y\}$ , and null otherwise. Then  $G', \eta'$  satisfy the conclusion of the lemma and (4) follows.

- (5) *If  $\{u, v\} = \{k_3, l_3\}$ , then the lemma holds.*

This follows immediately from (3) and (4).

- (6) *If  $u \in \{u_2, k_2\}$  and  $x, y \notin \{k_1, v_2, l_2, u_3, l_3, k_4, l_4, v''_2\}$  then the lemma holds.*

To prove (6) let  $G'$  and  $\eta'$  be obtained from  $\eta_0$  by first rerouting  $\eta_0(u_2u_3)$  along  $\eta_5(k_3l_3)$ , then routing the first new edge along  $\eta_5(k_1k_4) \cup \eta_5(k_4l_4) \cup \eta_5(l_4l_2)$ , and then routing the second new edge along  $\eta_5(k_5l_5) \cup Q$ , where  $Q$  is  $\eta_5(k_5u_2)$  if  $v = u_3$ ,  $\eta_5(k_5k_2)$  if  $v = l_4$ , and null otherwise. Then  $G', \eta'$  satisfy the conclusion of the lemma, because  $G'$  is a long 2-extension of  $G$ . This proves (6).

(7) *If  $v_3$  and  $v_2''$  are adjacent, then the lemma holds.*

To prove (7) let  $G', \eta'$  be obtained from  $\eta_0$  by

- first rerouting  $\eta_4(u_2u_3)$  along  $\eta_4(k_3l_3)$ ,
- then rerouting  $\eta_4(k_2u_2) \cup \eta_4(u_2v_2)$  along  $\eta_2(k_2l_2)$ ,
- then rerouting  $\eta_4(u_1v_1)$  along  $\eta_1(k_1l_1)$ ,
- then rerouting  $\eta_4(k_1k_3)$  along  $\eta_4(k_4l_4)$ ,
- then rerouting  $\eta_4(v_3v_2'')$  along  $\eta_4(u_3u_2) \cup \eta_4(u_2v_2)$ ,
- then routing the first new edge along  $\eta_4(u_2k_2)$ ,
- then routing the second new edge along  $\eta_4(k_1k_3)$ ,
- and finally routing the third new edge along  $\eta_4(u_1v_1)$ .

Then  $G'$  is a type E expansion of  $G$ , and hence the pair  $G', \eta'$  satisfies the conclusion of the lemma. This proves (7).

(8) *If  $\{u, v\}$  is one of  $\{u_2, u_3\}$ ,  $\{u_3, l_3\}$ ,  $\{l_3, u_4\}$  or  $\{u_1, u_4\}$ , then the lemma holds.*

To prove (8) we may assume by (2) that  $\{u, v\} = \{u_2, u_3\}$  or  $\{u_3, l_3\}$ . Assume first that  $v_3, v_4 \notin \{x, y\}$ . Let  $G', \eta'$  be obtained from  $\eta_0$  by

- first rerouting  $\eta_4(u_1u_4)$  along  $\eta_4(l_3k_3)$ ,
- then rerouting  $\eta_4(k_3k_1) \cup \eta_4(k_1u_1) \cup \eta_4(u_1v_1)$  along  $\eta_4(k_2l_4) \cup \eta_4(l_4k_4) \cup \eta_4(k_4l_1)$ ,
- then rerouting  $\eta_4(l_1v_1')$  along  $\eta_4(l_4l_2)$  and
- finally routing the new edge along  $\eta_5(k_5l_5) \cup Q$ , where  $Q$  is either null, or a path of  $\eta_4(G_4)$  with one end  $\eta_5(l_5)$ , the other end in  $\eta'(v_1v_1'')$ , and otherwise disjoint from  $\eta'(G)$ .

The graph  $G'$  is a long extension of  $G$ , unless  $\{u, v\} = \{u_3, l_3\}$  and  $\{x, y\} = \{k_2, l_4\}$ , in which case (8) follows from (4). Thus (8) holds if  $v_3, v_4 \notin \{x, y\}$ , and so we may assume that either  $x = v_3$  or  $x = v_4$ . As a second case assume that  $x = v_4$ . If  $\{u, v\} = \{u_2, u_3\}$ , then (8) follows from (6), and so let  $\{u, v\} = \{u_3, l_3\}$ . Let  $G'$  be obtained from  $G_4$  by deleting the edges  $k_2l_4$  and  $k_4l_1$  and suppressing degree two vertices. Then  $G'$  is isomorphic to a type E expansion of  $G$ , and so (8) follows. This completes the second case. Thirdly, let  $x = v_3$ . Since the cases  $\{u, v\} = \{u_2, u_3\}$  and  $\{u, v\} = \{u_3, l_3\}$  are symmetric by (4.2), we may assume that  $\{u, v\} = \{u_2, u_3\}$ . If  $v_3$  and  $v_2''$  are adjacent, then (8) follows from

(7); otherwise it follows from (6). This proves (8).

(9) *If  $u = k_3$  then the lemma holds.*

To prove (9) let  $u = k_3$ . By (5) we may assume that  $v \neq l_3$  (and hence  $\{x, y\} \neq \{k_4, l_4\}$ ), by (3) we may assume that  $\{x, y\} \cap R \neq \emptyset$ , and by (2) we may assume that  $\{x, y\} \neq \{u_2, v_2\}$  and  $\{x, y\} \neq \{l_1, k_4\}$ . By (8) we may assume that  $\{x, y\} \neq \{u_2, u_3\}$  and  $\{x, y\} \neq \{u_1, u_4\}$ . Let  $G', \eta'$  be obtained from  $\eta_0$  first by rerouting  $\eta_4(k_1k_3) \cup \eta_4(k_3k_2)$  along  $\eta_4(k_1k_4) \cup \eta_4(k_4l_4) \cup \eta_4(l_4k_2)$ , then rerouting  $\eta_4(u_1v_1)$  along  $\eta_4(k_4l_1)$ , and finally routing the new edge along  $\eta_4(l_3k_3) \cup \eta_5(k_3k_5) \cup \eta_5(k_5l_5) \cup Q$ , where  $Q$  is either null or  $\eta_5(l_2l_5)$  or  $\eta_5(v_1l_5)$ . If  $\{x, y\} \neq \{u_4, v_4\}$  and  $\{x, y\} \neq \{u_3, v_3\}$  then  $G'$  is a long extension of  $G$ , and hence the lemma holds. From the symmetry we may assume that  $\{x, y\} = \{u_4, v_4\}$ . If  $\{u, v\} = \{k_3, k_2\}$  then (9) follows from (6), and so we may assume that  $\{u, v\} = \{k_1, k_3\}$ . Let  $G', \eta'$  be obtained from  $\eta_0$  by first rerouting  $\eta_4(u_1u_4)$  along  $\eta_5(k_5l_5)$ , then rerouting  $\eta_4(k_1u_1) \cup \eta_4(u_1v_1)$  along  $\eta_4(k_1k_4) \cup \eta_4(k_4l_1)$ , and then routing the new edge along  $\eta_4(u_4u_1) \cup \eta_4(u_1v_1)$ . Then  $G', \eta'$  satisfy the conclusion of the lemma, and hence (9) holds.

(10) *If  $u \in \{l_1, k_4\}$  and  $x = v_2$  then the lemma holds.*

To prove (10) we first define two paths  $Q, Q'$ . Let  $Q$  be the path of  $\eta_4(v_2l_2) \cup \eta_4(v_2u_2) \cup \eta_4(v_2v_2'')$  with one end  $\eta_5(l_5)$  and the other end in  $\eta_4(v_2u_2) \cup \eta_4(v_2v_2'')$ , and let  $Q'$  be the path of  $\eta_1(k_1l_1) \cup \eta_0(v_1v_1') \cup \eta_4(k_4l_4)$  with one end  $\eta_5(k_5)$  and the other end in  $\eta_0(v_1v_1')$ . Let  $G' = G + (u_1, u_2, v_1', w)$  and  $\eta'$  be obtained from  $\eta_0$  by rerouting an appropriate path along  $Q \cup \eta_5(l_5k_5) \cup Q'$ , and then routing the new edge along  $\eta_2(k_2l_2) \cup \eta_2(l_2v_1')$ . Then  $G', \eta'$  satisfy the conclusion of the lemma, thus proving (10).

(11) *If  $\{u, v\}$  equals one of  $\{u_2, v_2\}$ ,  $\{u_1, v_1\}$ ,  $\{u_2, k_2\}$  or  $\{u_1, k_1\}$ , then the lemma holds.*

To prove (11) we may assume by (2) that  $\{u, v\} = \{u_2, v_2\}$  or  $\{u, v\} = \{u_2, k_2\}$ . If  $\{x, y\} = \{u_4, v_4\}$  or  $\{x, y\} = \{v_3, v_3'\}$ , where  $v_3' \neq u_3$  is a neighbor of  $v_3$ , then (11) follows from (6) and (7). If  $\{x, y\} = \{k_1, k_3\}$  then (11) follows from (9), and if  $\{x, y\} = \{k_4, l_4\}$ , then (11) follows from (10). We may therefore assume that none of the above hold. Let  $G', \eta'$  be obtained from  $\eta_0$  by first rerouting  $\eta_4(u_1u_4)$  along  $\eta_4(k_3l_3)$ , then rerouting  $\eta_4(v_1l_1)$  along  $\eta_1(k_1l_1)$ , then rerouting  $\eta_4(u_2v_2)$  along  $\eta_2(k_2l_2)$ , then rerouting  $\eta_4(l_2v_1')$

along  $\eta_4(k_4l_4)$ , and finally routing the new edge along  $Q \cup \eta_5(k_5l_5) \cup Q'$ , where  $Q$  is either null or  $\eta_5(k_5u_2)$ , and  $Q'$  is either null or a subpath of  $\eta_4(l_2v'_1) \cup \eta_4(u_4u_1) \cup \eta_4(v_1l_1)$  with one end  $\eta_5(l_5)$  and the other end in  $\{\eta_4(v'_1), \eta_4(u_1), \eta_4(v_1)\}$ . Then the graph  $G'$  is a long extension of  $G$ , and hence (11) holds.

(12) *If  $u \in \{l_1, k_4\}$ ,  $x \in V(G) - (R \cup \{v''_1, w\})$  and  $y \neq v''_1$ , then the lemma holds.*

To prove (12) let  $G', \eta'$  be obtained from  $\eta_0$  first by rerouting  $\eta_4(u_4u_1)$  along  $\eta_4(k_3l_3)$ , then rerouting  $\eta_4(k_1k_3)$  along  $\eta_4(k_2l_4) \cup \eta_4(l_4k_4) \cup \eta_4(k_4k_1)$ , then by rerouting  $\eta_0(v_1v'_1)$  along  $\eta_4(l_4l_2)$ , then routing a first new edge along  $\eta_4(k_1k_3)$ , and finally routing a second new edge along  $Q \cup \eta_5(k_5l_5)$ , where  $Q$  is a suitable path of  $\eta_4(G_4)$  with one end  $\eta_5(k_5)$  and the other end in  $\eta_4(k_1k_4) \cup \eta_4(k_4l_4)$ . Then  $G', \eta'$  satisfy the conclusion of (12) thus proving (12).

(13) *If  $\{u, v\} = \{k_4, l_4\}$  then the lemma holds.*

This follows from (2), (3), (5), (8), (10), (11) and (12).

(14) *If  $\{u, v\}$  equals one of  $\{k_1, k_4\}$ ,  $\{k_4, l_1\}$ ,  $\{k_2, l_4\}$  or  $\{l_4, l_2\}$  then the lemma holds.*

To prove (14) we may assume by (2) that  $\{u, v\} = \{k_1, k_4\}$  or  $\{u, v\} = \{k_4, l_1\}$ . By (3), (5), (8), (10), (11) and (12) we may assume that  $\{u, v\} = \{k_1, k_4\}$ , and  $\{x, y\} = \{l_2, v'_1\}$  or  $\{x, y\} = \{v_1, v''_1\}$ . Let  $\{u, v\} = \{k_1, k_4\}$ , and assume first that  $\{x, y\} = \{l_2, v'_1\}$ . Let  $G', \eta'$  be obtained from  $\eta_0$  first by rerouting  $\eta_0(v'_1l_2)$  along  $\eta_4(l_1k_4) \cup \eta_4(k_4l_4) \cup \eta_4(l_4l_2)$ , and then routing the new edge along  $\eta_5(k_1k_5) \cup \eta_5(k_5l_5) \cup \eta_5(l_5v'_1)$ . Then  $G', \eta'$  satisfy the conclusion of the lemma. We may therefore assume that  $\{x, y\} = \{v_1, v''_1\}$ . In this case let  $G', \eta'$  be obtained from  $\eta_0$  by routing the first new edge along  $\eta_4(k_1k_4) \cup \eta_4(k_4l_4) \cup \eta_4(l_4l_2)$ , and routing the second new edge along  $\eta_5(k_5l_5)$ . Then  $G'$  is a long 2-extension of  $G$  by (2.1) and (2.5) (or by (1)), and hence the pair  $G', \eta'$  satisfies the conclusion of the lemma. This proves (14).

(15) *If  $\{u, v\} = \{v_2, l_2\}$  or  $\{u, v\} = \{v_1, l_1\}$  then the lemma holds.*

To prove (15) we may assume by (2) that  $\{u, v\} = \{v_1, l_1\}$ . By (5), (8), (9), (10), (11) and (12) we may assume that  $\{x, y\} = \{v''_1, z\}$ , where  $z \neq v_1$  is a neighbor of  $v''_1$ . But then

$G_5$  is isomorphic to  $G_4 + (u_1, v_1, v_1'', z')$ , where  $z' \notin \{v_1, z\}$  is the third neighbor of  $v_1''$ , and hence (15) follows from (11).

(16) *If  $\{u, v\} = \{l_2, v_1'\}$  or  $\{u, v\} = \{l_1, v_1'\}$  then the lemma holds.*

To prove (16) we may assume by (2) that  $\{u, v\} = \{l_2, v_1'\}$ . By (2), (5), (8), (9), (11), (12) and (14) we may assume that  $\{x, y\} = \{v_2'', z\}$ , where  $z \neq v_2$  is a neighbor of  $v_2''$ . Let  $G', \eta'$  be obtained from  $\eta_0$  by first rerouting  $\eta_4(v_2v_2'')$  along  $\eta_5(k_5l_5)$ , then rerouting  $\eta_4(u_2v_2) \cup \eta_4(v_2l_2)$  along  $\eta_2(k_2l_2)$ , and finally routing the new edge along  $\eta_4(u_2v_2) \cup \eta_4(v_2v_2'')$ . If  $v_2''$  is not adjacent to  $v_3$ , then  $G'$  is a long extension of  $G$  by (2.1), and hence the pair  $G', \eta'$  satisfies the conclusion of the lemma. On the other hand if  $v_3$  and  $v_2''$  are adjacent, then (16) follows from (7). This completes the proof of (16).

The lemma now follows from (5), (8), (9), (11), (13), (14), (15) and (16).  $\square$

**(5.4)** *Let  $G, G_4, H$  be cubic graphs, let  $C$  be a quadrangle in  $G$ , let  $F$  be a graph of minimum degree at least two, let  $G$  be quad-connected, let  $G_4$  be a type  $F$  expansion of  $G$  based at  $C$  such that its core is disjoint from  $F$ , let  $\eta : G_4 \hookrightarrow H$  fix  $F$ , and let  $H$  be dodecahedrally connected. Then there exist an integer  $n \in \{1, 2, 3\}$ , a long  $n$ -extension  $G'$  of  $G$  based at  $C$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow H$  that fixes  $F$ .*

*Proof.* This follows immediately from (5.2) and (5.3).  $\square$

**(5.5)** *Let  $G, G_5, H$  be cubic graphs, let  $C$  be a quadrangle in  $G$ , let  $F$  be a graph of minimum degree at least two, let  $G$  be quad-connected, let  $G_5$  be a type  $G$  or  $H$  expansion of  $G$  based at  $C$ , let  $F$  be a subgraph of both  $G$  and  $G_5$ , and let  $H$  be dodecahedrally connected. Then there exist an integer  $n \in \{1, 2, 3\}$ , a long  $n$ -extension  $G'$  of  $G$  based at  $C$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow H$  that fixes  $F$ .*

*Proof.* Let  $G_1$  be a short 1-extension of  $G$  such that  $G_5$  is a type  $F$  expansion of  $G_1$  based at the new quadrangle of  $G_1$ . By (5.4) applied to  $G_1$  and the new quadrangle of  $G_1$  there exist an integer  $k \in \{1, 2, 3\}$ , a long  $k$ -extension  $G_2$  of  $G_1$ , and a homeomorphic embedding  $G_2 \hookrightarrow H$ . Then  $G_2$  is a long  $(k+1)$ -extension of  $G$  based at  $C$ , and so if  $k \leq 2$ , then the lemma holds. We may therefore assume that  $k = 3$ . By (4.5) we may assume that there exist a type  $F$  expansion  $G_3$  of  $G$  based at  $C$  and a homeomorphic embedding  $G_3 \hookrightarrow H$

that fixes  $F$ . The conclusion of the lemma now follows from (5.4) applied to the graph  $G$  and quadrangle  $C$ .  $\square$

**(5.6)** *Let  $G, H$  be cubic graphs, let  $C$  be a quadrangle in  $G$ , let  $F$  be a graph of minimum degree at least two disjoint from  $C$ , let  $\eta : G \hookrightarrow H$  fix  $F$ , let  $G$  be quad-connected, and let  $H$  be dodecahedrally connected. Then there exist an expansion  $G'$  of  $G$  of type A, B, C, D, or E based at  $C$ , and a homeomorphic embedding  $G' \hookrightarrow H$  that fixes  $F$ .*

*Proof.* By (4.9) there exist an expansion  $G_1$  of  $G$  of type A, B, C, D, E, F, G or H and a homeomorphic embedding  $\eta_1 : G_1 \hookrightarrow H$  that fixes  $F$ . We may assume that  $G_1$  is of type F, G, or H, for otherwise  $G_1, \eta_1$  satisfy the theorem. By (5.4) and (5.5) applied to  $G, G_1, H$  and  $\eta_1$  there exist an integer  $n \in \{1, 2, 3\}$ , a long  $n$ -extension  $G_2$  of  $G$  based at  $C$  and a homeomorphic embedding  $\eta_2 : G_2 \hookrightarrow H$  that fixes  $F$ . By (4.3) and (4.4) there exist an expansion  $G_3$  of  $G$  of type A, B, C, D, or E and a homeomorphic embedding  $\eta_3 : G_3 \hookrightarrow H$  that fixes  $F$ , as desired.  $\square$

## 6. A TWO-EXTENSION THEOREM

In this section we prove a preliminary weaker version of (1.3). In (6.1) we prove it when  $H$  is dodecahedrally connected, and in (6.2) we prove it for cyclically 5-connected graphs  $H$ .

**(6.1)** *Let  $G, H$  be cubic graphs, let  $G$  be cyclically 5-connected, let  $H$  be dodecahedrally connected, and let  $\eta : G \hookrightarrow H$  be a homeomorphic embedding. Then there exist a cyclically 5-connected cubic graph  $G'$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow H$  such that  $G'$  is a 1- or 2-extension of  $G$ .*

*Proof.* Let  $G, H, \eta$  be as stated. By (3.4) there exist a 1-extension  $G_0 = G + (u_2, v_3, u_1, v_4)$  of  $G$  and a homeomorphic embedding  $\eta_0 : G_0 \hookrightarrow H$ . Let  $u_3, u_4$  be the new vertices of  $G_0$ . If  $G_0$  is cyclically 5-connected, then  $G_0, \eta_0$  satisfy the conclusion of (6.1), and so we may assume that  $G_0$  is not cyclically 5-connected. By (2.1) we may assume that say  $u_1$  is adjacent to  $u_2$ . Then  $G_0$  is quad-connected, and has a unique quadrangle  $C_0$ , where  $V(C_0) = \{u_1, u_2, u_3, u_4\}$ . By (5.6) there exist an expansion  $G_2$  of  $G_0$  of type A, B, C, D or E based  $C_0$ , and a homeomorphic embedding  $\eta_2 : G_2 \hookrightarrow H$ . If  $G_2$  is of type A, then

the pair  $G_2, \eta_2$  satisfies the conclusion of the lemma, and so it remains to consider types B, C, D and E.

Let us assume now that  $G_2$  is of type B, C or D, and let  $G_1, G_2$  be a standard generating sequence for  $G_2$ . Let  $G_1 = G_0 + (a_1, a_2, v_1, w)$ , where  $a_1, a_2 \in V(C_0)$ ,  $v_1, w \notin V(C_0)$ , and let  $k_1, l_1$  be the new vertices of  $G_1$ . Since  $G_1$  is a short extension of  $G_0$ , by (2.1) we may assume that say  $a_1$  is adjacent to  $v_1$ , and hence  $G_1$  has a unique quadrangle, say  $C_1$ , and its vertex-set is  $\{a_1, v_1, k_1, l_1\}$ . Let  $\xi : G \hookrightarrow G_2$  be the canonical homeomorphic embedding determined by the generating sequence  $G_0, G_1, G_2$ , let  $\zeta_0 = \xi \circ \eta_2$ , let  $\zeta_1$  be obtained from  $\zeta_0$  by rerouting  $\zeta_0(u_1 u_2)$  along  $\eta_2(u_3 u_4)$ , and let  $\zeta_2$  be obtained from  $\zeta_0$  by rerouting  $\eta_2(a_1 v_1)$  along  $\eta_2(k_1 l_1)$ .

(1) *We may assume that  $\{a_1, a_2\} = \{u_1, u_2\}$ .*

To prove (1) we first notice that by (4.2) we may assume that  $\{a_1, a_2\} = \{u_1, u_2\}$  or  $\{a_1, a_2\} = \{u_3, u_4\}$ . But if  $\{a_1, a_2\} = \{u_3, u_4\}$ , then by replacing  $\eta$  by  $\zeta_1$  we can arrange that (1) holds.

From the symmetry between  $u_1$  and  $u_2$  we may assume that  $a_1 = u_1$  and  $a_2 = u_2$ . Then  $v_1$  is the neighbor of  $u_1$  in  $G_0$  that does not belong to  $C_0$ . Let  $v_2$  be the neighbor of  $u_2$  in  $G_0$  that does not belong to  $C_0$ .

(2) *We may assume that  $w$  and  $v_2$  are adjacent in  $G$ .*

To prove (2) suppose that  $w$  and  $v_2$  are not adjacent, and let  $G' = G + (u_2, v_2, v_1, w)$  and  $\eta'$  be obtained from  $\zeta_1$  by routing the new edge along  $\eta_1(u_2 k_1) \cup \eta_1(k_1 l_1)$ . Since  $G + (u_2, v_2, v_1, w)$  is cyclically 5-connected by (2.3), (2) holds.

By (2)  $G$  has a circuit with vertex-set  $\{u_1, u_2, v_2, w, v_1\}$ . Let  $v'_1$  be the neighbor of  $v_1$  not on this circuit, and let  $v'_2$  be defined similarly. We distinguish cases depending on the type of the expansion  $G_2$ .

Let us assume first that  $G_2$  is a type B expansion of  $G_0$ . Then  $G_2 = G_1 + (v_1, l_1, x, y)$  for some  $x, y \in V(G_1)$ . Let  $k_2, l_2$  be the new vertices of  $G_2$ . Let us assume first that  $\{x, y\} = \{u_4, v_4\}$ . Let  $G' = G + (u_2, v_2, v_1, v'_1)$  and  $\eta'$  be obtained from  $\zeta_1$  by first rerouting  $\zeta_1(v_1 u_1)$  along  $\eta_2(k_2 l_2)$ , and then routing the new edge along  $\eta_1(u_1 u_2) \cup \eta_1(u_1 v_1)$ . Since  $G + (u_2, v_2, v_1, v'_1)$  is cyclically 5-connected by (2.5), the pair  $G', \eta'$  satisfies the conclusion

of the theorem, as required. We may therefore assume that  $\{x, y\} \neq \{u_4, v_4\}$ . Let  $G', \eta'$  be obtained from  $\zeta_0$  by routing the first new edge along  $\eta_1(k_1l_1)$ , and then routing the second new edge along  $\eta_2(k_2l_2)$  (or along  $\eta_2(k_2l_2) \cup \eta_2(l_2u_3)$  if  $\{x, y\} = \{u_3, u_4\}$ ). Then  $G', \eta'$  satisfy the conclusion of the theorem. This completes the case when  $G_2$  is a type B expansion of  $G_0$ .

We now assume that  $G_2$  is a type C expansion of  $G_0$ . Since  $G_1 + (k_1, l_1, v_2, w)$  is not cyclically 5-connected, there are only two cases to consider. Assume first that  $G_2 = G_1 + (k_1, l_1, v_2, v'_2)$ , and let  $k_2, l_2$  be the new vertices of  $G_2$ . Let  $G' = G + (u_1, v_1, v_2, v'_2)$  and  $\eta'$  be obtained from  $\zeta_2$  by routing the new edge along  $\eta_2(k_2l_2)$ . Since  $G'$  is cyclically 5-connected by (2.5) the theorem holds. Secondly, let us assume that  $G_2 = G_1 + (u_1, v_1, v_4, v'_4)$ , where  $v'_4 \neq u_4$  is a neighbor of  $v_4$  in  $G$ , and let  $k_2, l_2$  be the new vertices of  $G_2$ . Let  $G' = G + (v_1, v'_1, v_4, v'_4)$  and  $\eta'$  be obtained from  $\zeta_2$  by routing the new edge along  $\eta_2(v_1k_2) \cup \eta_2(k_2l_2)$ . If  $G'$  is cyclically 5-connected, then the pair  $G', \eta'$  is as desired. We may therefore assume that  $G + (v_1, v'_1, v_4, v'_4)$  is not cyclically 5-connected, and hence  $v'_1$  and  $v'_4$  are adjacent by (2.3). Let  $G'$  and  $\eta'$  be obtained from  $\zeta_2$  by first rerouting  $\zeta_2(v'_1v'_4)$  along  $\eta_2(v_1k_2) \cup \eta_2(k_2l_2)$ , then routing a first new edge along  $\eta_1(u_3u_4)$  and then routing a second new edge along  $\eta_2(k_2u_1)$ . Then  $G'$  is isomorphic to  $G_0 + (u_1, u_4, v'_1, v'_4)$ . Since  $G_0 + (u_1, u_4, v'_1, v'_4)$  is cyclically 5-connected by (2.3), the pair  $G', \eta'$  is as desired. This completes the case when  $G_2$  is a type C expansion.

We now assume that  $G_2$  is a type D expansion of  $G_0$ ; then  $G_2 = G_1 + (k_1, u_1, u_3, v_3)$ . Let  $k_2, l_2$  be the new vertices of  $G_2$ . Let  $G' = G + (v_1, w, u_2, v_3)$  and  $\eta'$  be obtained from  $\zeta_1$  by routing the new edge along  $\eta_2(k_2l_2) \cup \eta_2(k_2k_1) \cup \eta_2(k_1l_1)$ . Since  $G'$  is cyclically 5-connected by (2.5) the theorem holds in this case. This completes the case that  $G_2$  is a type D expansion.

Finally we assume that  $G_2$  is a type E expansion of  $G_0$ . Let  $G_1, G'_2, G_2$  be a standard generating sequence for  $G_2$ . From the symmetry we may assume that  $G'_2 = G_1 + (u_3, u_4, v_3, v'_3)$ , where  $v'_3 \neq u_3$  is a neighbor of  $v_3$ , and  $G_2 = G'_2 + (k'_2, u_3, k_1, u_1)$ , where  $k'_2, l'_2$  are the new vertices of  $G'_2$ . Let  $k_2, l_2$  be the new vertices of  $G_2$ . Let  $G'$  and  $\eta'$  be obtained from  $\zeta_0$  by routing the first new edge along  $\eta_2(l_2k_2) \cup \eta_2(k_2k'_2) \cup \eta_2(k'_2l'_2)$  and then routing the second new edge along  $\eta_2(k_1l_1)$ . Since  $G'$  is cyclically 5-connected by (2.4),



the theorem holds in this case. This completes the case when  $G_2$  is a type E expansion of  $G_0$ , and hence the proof of the theorem.  $\square$

Let us recall that circuit expansion was defined prior to (1.3).

**(6.2)** *Let  $G, H$  be non-isomorphic cyclically 5-connected cubic graphs, and let  $\eta : G \hookrightarrow H$  be a homeomorphic embedding. Then there exist a cyclically 5-connected cubic graph  $G'$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow H$  such that  $G'$  is either a 1- or 2-extension or a circuit expansion of  $G$ .*

*Proof.* Let  $G, H, \eta$  be as stated. By (3.4) there exist a 1-extension  $G_0 = G + (u_2, v_3, u_1, v_4)$  of  $G$  and a homeomorphic embedding  $\eta_0 : G_0 \hookrightarrow H$ . Let  $u_3, u_4$  be the new vertices of  $G_0$ . If  $G_0$  is cyclically 5-connected, then  $G_0, \eta_0$  satisfy the conclusion of (6.1), and so we may assume that  $G_0$  is not cyclically 5-connected. By (2.1) we may assume that say  $u_1$  is adjacent to  $u_2$ . Then  $G_0$  is quad-connected, and has a unique quadrangle  $C_0$ , where  $V(C_0) = \{u_1, u_2, u_3, u_4\}$ . By (4.10) there exist an expansion  $G_2$  of  $G_0$  of type A, B, C, D, E, F, G or H based at  $C_0$ , and a homeomorphic embedding  $\eta_2 : G_2 \hookrightarrow H$ . If  $G_2$  is an expansion of type A, B, C, D or E then the theorem holds by the proof of (6.1). We may therefore assume that  $G_2$  is an expansion of type F, G or H. Let  $\zeta_1$  be defined as in the proof of (6.1).

Assume first that  $G_2$  is an expansion of type F. Since  $G$  is cyclically 5-connected,  $v_1$  and  $v_4$  have no common neighbor in  $G$ , and similarly  $v_2$  and  $v_3$  have no common neighbor in  $G$ . Therefore  $G_2$  is based on either  $u_1u_2$ , or  $u_3u_4$ . In either case  $G_2$  is a circuit expansion of  $G$ , and so the pair  $G_2, \eta_2$  satisfies the conclusion of the theorem.

Secondly, let us assume that  $G_2$  is an expansion of type G. Let  $G_1$  be a short 1-extension of  $G_0$  based at  $C_0$  such that  $G_2$  is a type F expansion of  $G_1$ , and let  $C_1$  be the unique quadrangle of  $G_1$ . By replacing  $\eta$  by  $\zeta_1$  and by using symmetry we may assume that  $G_1 = G_0 + (u_1, u_2, v_1, w)$ , where  $w \neq u_1$  is a neighbor of  $v_1$ . Let  $k_1, l_1$  be the new vertices of  $G_1$ ; then the vertex-set of  $C_1$  is  $\{u_1, v_1, l_1, k_1\}$ . From claim (1) in the proof of (6.1) we may assume that  $w$  and  $v_2$  are adjacent. Let  $G'$  be obtained from  $G_2$  by deleting the edge  $v_2w$  and suppressing the resulting vertices of degree two, and let  $\eta'$  be the restriction of  $\eta_2$  to  $G'$ . Then  $G'$  is isomorphic to a circuit expansion of  $G$ , and so the theorem holds.

Finally let us assume that  $G_2$  is an expansion of type H. Using the same symmetry as before we may assume that  $G_0$  has a quadrangle  $D$  with vertex-set  $\{x_1, x_2, x_3, x_4\}$ , where  $u_1$  is adjacent to  $x_1$ , the vertices  $u_2$  and  $x_2$  have a common neighbor, and  $u_4$  and  $x_4$  have a common neighbor, say  $z$ . Then the set  $V(D) \cup \{u_1, z\}$  violates the dodecahedral connectivity of  $G$ . This completes the case when  $G_2$  is a type H expansion, and hence a proof of the theorem.  $\square$

## 7. A ONE-EXTENSION THEOREM

In this section we prove (1.3) and (1.4).

**(7.1)** *Let  $G, H$  be cyclically 5-connected cubic graphs, let  $u_1, u_2, u_3, u_4, u_5$  (in order) be the vertices of a path of  $G$ , let  $G_2 = G + (u_1, u_2, u_3, u_4) + (u_2, u_3, u_4, u_5)$ , and let  $\eta_2 : G_2 \hookrightarrow H$ . Then there exist a cyclically 5-connected handle expansion  $G'$  of  $G$  and a homeomorphic embedding  $G' \hookrightarrow H$ .*

*Proof.* Let  $v_2 \notin \{u_1, u_3\}$  be the third neighbor of  $u_2$ , and let  $v_3$  and  $v_4$  be defined similarly. Let  $G_1 = G + (u_1, u_2, u_3, u_4)$ , let  $k_1, l_1$  be the new vertices of  $G_1$ , let  $k_2, l_2$  be the new vertices of  $G_1 + (u_2, u_3, u_4, u_5)$ , and let  $\eta$  be the restriction of  $\eta_2$  to  $G$ . Let  $\zeta_1$  be obtained from  $\eta$  by rerouting  $\eta_2(u_2u_3)$  along  $\eta_2(k_1l_1)$ . By considering the path  $\eta_2(l_2k_2) \cup \eta_2(k_2u_2)$  we can extend  $\zeta_1$  to a homeomorphic embedding  $G + (v_2, u_2, u_4, u_5) \hookrightarrow H$ . We deduce that if  $G + (v_2, u_2, u_4, u_5)$  is cyclically 5-connected, then the lemma holds. Thus we may assume that that is not the case, and hence  $v_2$  and  $u_5$  are adjacent in  $G$  by (2.3).

Let  $G' = G + (u_3, v_3, u_5, v_2)$  and  $\eta'$  be obtained from  $\zeta_1$  by first rerouting  $\eta_2(v_2u_5)$  along  $\eta_2(k_2l_2) \cup \eta_2(k_2u_2)$ , and then by routing the new edge along  $\eta_2(k_2u_3)$ . Since  $G'$  is cyclically 5-connected by (2.5), the lemma follows.  $\square$

**(7.2)** *Let  $G$  be a cyclically 5-connected cubic graph, and let  $\dots, u_{-1}, u_0, u_1, \dots$  and  $\dots, v_{-1}, v_0, v_1, \dots$  be two doubly infinite sequences of (not necessarily distinct) vertices of  $G$  such that for all integers  $i$ , the neighbors of  $u_i$  are  $u_{i-1}$ ,  $u_{i+1}$  and  $v_i$ , and the neighbors of  $v_i$  are  $v_{i-2}$ ,  $v_{i+2}$  and  $u_i$ . Then there exists an integer  $p \geq 5$  ( $p \geq 10$  if  $p$  is even) such that  $u_i = u_{i+p}$  and  $v_i = v_{i+p}$  for all integers  $i$ , and the vertices  $u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_p$  are pairwise distinct. Thus  $G$  is a biladder.*

*Proof.* Choose  $p > 0$  minimum such that for some integer  $i$ , one of  $u_i, v_i$  equals one of  $u_{i+p}, v_{i+p}$ . Suppose first that  $u_i = v_{i+p}$ . Then  $p > 2$ , and the neighborhood set of  $u_i$  equals the neighborhood set of  $v_{i+p}$ , so one of  $u_{i-1}, u_{i+1}, v_i$  equals  $v_{i+p-2}$ , contrary to the choice of  $p$ . If  $v_i = u_{i+p}$ , then similarly one of  $u_{i+p-1}, u_{i+p+1}, v_{i+p}$  equals  $v_{i+2}$ , again contrary to the choice of  $p$ .

So either  $u_i = u_{i+p}$  or  $v_i = v_{i+p}$ ; and then as before, it follows that  $u_i = u_{i+p}$  and  $v_i = v_{i+p}$  for all integers  $i$ . It follows from the choice of  $p$  that the vertices  $u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_p$  are pairwise distinct.  $\square$

**(7.3)** *Let  $G, H$  be cyclically 5-connected cubic graphs, let  $H$  be a long 2-extension of  $G$ , and assume that there does not exist a handle expansion  $G'$  of  $G$  which admits a homeomorphic embedding  $G' \hookrightarrow H$ . Then both  $G, H$  are biladders.*

*Proof.* Since  $H$  is a 2-extension of  $G$ , there exist vertices  $v_1, v_3, u_3, u_2$  of  $G$  and vertices  $a_1, a_2, a_3, a_4$  of  $G_1 = G + (v_1, v_3, u_3, u_2)$  such that  $H = G_1 + (a_1, a_2, a_3, a_4)$ . Let  $k_1, l_1$  be the new vertices of  $G_1$ . Then  $G_1$  is not cyclically 5-connected, and so by (2.1) we may assume that  $v_3$  is adjacent to  $u_3$  in  $G$ . Thus  $G_1$  is quad-connected and has a unique quadrangle  $C_1$ , where  $C_1$  has vertex-set  $v_3, u_3, l_1, k_1$ . Furthermore, one of  $\{a_1, a_2\}, \{a_3, a_4\}$  is equal to one of  $\{v_3, u_3\}, \{u_3, l_1\}, \{l_1, k_1\}$  or  $\{k_1, v_3\}$ . From the symmetry (and making use of the homeomorphic embedding obtained from the canonical homeomorphic embedding  $G \hookrightarrow H$  by rerouting  $v_3 u_3$  along  $k_1 l_1$ ) we may assume that either  $\{a_1, a_2\} = \{v_3, u_3\}$ , or  $\{a_1, a_2\} = \{u_3, l_1\}$ . Let  $v_2 \neq u_3$  be a neighbor of  $u_2$  in  $G$ . In the former case, since  $G + (a_1, a_2, a_3, a_4)$  is not cyclically 5-connected, we may assume from the symmetry that  $\{a_3, a_4\} = \{u_2, v_2\}$ , in which case we obtain a contradiction from (7.1) applied to the path of  $G$  with vertex-set  $\{v_1, v_3, u_3, u_2, v_2\}$ .

We may therefore assume that  $\{a_1, a_2\} = \{u_3, l_1\}$ , and further (by replacing  $v_2$  if necessary) that  $\{a_3, a_4\} = \{v_2, v_4\}$ , where  $v_4 \neq u_2$  is a neighbor of  $v_2$ . Then  $v_1 \neq v_4$ , because  $G$  is cyclically 5-connected. Thus  $G$  has a path  $P$  with vertex-set  $v_1, v_3, u_3, u_2, v_2, v_4$  (in order) such that  $H = G \& (v_1, v_3, u_3, u_2, v_2, v_4)$ . (The  $\&$  operator was defined prior to (2.4)). Let  $u_1$  be the neighbor of  $u_2$  not on  $P$ , and let  $u_4$  be the neighbor of  $u_3$  not on  $P$ . Assume that for some integers  $m, n$  with  $m \leq 1$  and  $n \geq 4$  we have already constructed

(not necessarily distinct) vertices  $u_m, u_{m+1}, \dots, u_n, v_m, v_{m+1}, \dots, v_n$  of  $G$  such that for all  $i = m+1, m+2, \dots, n-1$

- (i)  $u_i$  is adjacent in  $G$  to  $u_{i+1}$  and  $u_m$  is adjacent in  $G$  to  $u_{m+1}$ ,
- (ii)  $u_i$  is adjacent in  $G$  to  $v_i$ ,
- (iii)  $v_{i-1}$  is adjacent in  $G$  to  $v_{i+1}$ ,
- (iv) there exists a homeomorphic embedding

$$\eta_n : G \& (v_{n-3}, v_{n-1}, u_{n-1}, u_{n-2}, v_{n-2}, v_n) \hookrightarrow H,$$

and

- (v) there exists a homeomorphic embedding

$$\eta_m : G \& (v_m, v_{m+2}, u_{m+2}, u_{m+1}, v_{m+1}, v_{m+3}) \hookrightarrow H.$$

We shall construct  $u_{m-1}, v_{m-1}, u_{n+1}, v_{n+1}, \eta_{m-1}, \eta_{n+1}$  such that (i)-(v) are satisfied for all  $i = m, m+1, \dots, n$ .

Let  $L = G \& (v_{n-3}, v_{n-1}, u_{n-1}, u_{n-2}, v_{n-2}, v_n)$ , and let  $k, l, k', l'$  be the new vertices of  $L$ . Let  $\eta'$  be obtained from the restriction of  $\eta_n$  to  $G$  by rerouting  $\eta_n(v_{n-1}u_{n-1})$  along  $\eta_n(kl)$ . By considering the path  $\eta_n(k'l')$  we can extend  $\eta'$  to a homeomorphic embedding  $\eta'' : G + (u_{n-1}, u_n, v_{n-2}, v_n) \hookrightarrow H$ . Since  $G + (u_{n-1}, u_n, v_{n-2}, v_n)$  is not cyclically 5-connected by hypothesis, we deduce from (2.3) that  $u_n, v_n$  are adjacent. Let  $u_{n+1} \notin \{u_{n-1}, v_n\}$  be the third neighbor of  $u_n$ , and let  $v_{n+1} \notin \{u_{n-1}, v_{n-3}\}$  be the third neighbor of  $v_{n-1}$ . By considering the homeomorphic embedding  $\eta''$  and the path  $\eta_n(u_{n-1}v_{n-1})$  we can construct a homeomorphic embedding  $\eta_{n+1} : G \& (v_{n-2}, v_n, u_n, u_{n-1}, v_{n-1}, v_{n+1}) \hookrightarrow H$ . The vertices  $u_{m-1}, v_{m-1}$  and homeomorphic embedding  $\eta_{m-1}$  are defined analogously.

This completes the definition of two doubly infinite sequences of vertices  $\dots u_{-1}, u_0, u_1, \dots$  and  $\dots v_{-1}, v_0, v_1, \dots$  of  $G$  such that (i), (ii), (iii) hold for all integers  $i$ . It follows from (7.2) that both  $G, H$  are biladders, as required.  $\square$

**(7.4)** Let  $G, G_1$  be biladders, where  $|V(G_1)| = |V(G)| + 4$  and  $|V(G)| \notin \{10, 20\}$ , and let  $G_2$  be a handle expansion of  $G_1$ . Then there exist a handle expansion  $G'$  of  $G$  and a homeomorphic embedding  $G' \hookrightarrow G_2$ .

*Proof.* Let us assume that the vertices of  $G_1$  are numbered  $u_0, u_1, \dots, u_{p+1}, v_0, v_1, \dots, v_{p-1}$ , as in the definition of biladder. The edges of the form  $u_i v_i$  will be called *rungs*. Let us say that two edges  $e, f$  in a graph are *diverse* if they share no end and no end of  $e$  is adjacent to an end of  $f$ . It follows by inspection that if  $e, f$  are two diverse edges of  $G_1$ , then there exist two consecutive rungs such that they are not equal to  $e, f$  and upon the deletion of the rungs and suppression of the resulting degree two vertices the edges (corresponding to)  $e, f$  remain diverse in the smaller biladder. Since deleting two consecutive rungs and suppressing the resulting degree two vertices produces a graph isomorphic to  $G$ , we deduce that the theorem holds.  $\square$

The following variation of (7.4) is easy to see.

**(7.5)** *Let  $G, G_1$  be biladders, where  $|V(G_1)| = |V(G)| + 4$ , and let  $G_2$  be a circuit expansion of  $G_1$ . Then there exist a circuit expansion  $G'$  of  $G$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow G_2$ .*

The following theorem implies (1.3) and (1.4).

**(7.6)** *Let  $G, H$  be non-isomorphic cyclically 5-connected cubic graphs, assume that  $H$  topologically contains  $G$ , and assume that not both  $G, H$  are biladders. Assume further that if  $G$  is isomorphic to the Petersen graph, then  $H$  does not topologically contain the biladder on 14 vertices, and if  $G$  is isomorphic to Dodecahedron, then  $H$  does not topologically contain the biladder on 24 vertices. Then there exist a cyclically 5-connected cubic graph  $G'$  and a homeomorphic embedding  $G' \hookrightarrow H$  such that  $G'$  is either a handle or circuit expansion of  $G$ . Moreover, if  $H$  is dodecahedrally connected, then  $G'$  can be chosen to be a handle expansion.*

*Proof.* We proceed by induction on  $|V(H)| - |V(G)|$ . Let  $G, H$  be as stated, and assume that the theorem holds for all pairs  $G', H'$  with  $|V(H')| - |V(G')| < |V(H)| - |V(G)|$ . By (6.2) there exist a cyclically 5-connected cubic graph  $G_1$  and a homeomorphic embedding  $G_1 \hookrightarrow H$  such that  $G_1$  is a 1- or 2-extension or a circuit expansion of  $G$ . If  $H$  is dodecahedrally connected, then by (6.1)  $G_1$  can be chosen to be a 1- or 2-extension of  $G$ . We may assume that  $G_1$  is a 2-extension of  $G$ , for otherwise the conclusion of the theorem is

satisfied. From (7.3) we deduce that either the conclusion of the theorem is satisfied, or both  $G, G_1$  are biladders, and so we may assume the latter. Thus  $|V(G)| \neq 20$  by the hypothesis of the theorem. By the induction hypothesis applied to the pair  $G_1, H$  we deduce that there exist a handle or circuit expansion  $G_2$  of  $G_1$  and a homeomorphic embedding  $G_2 \hookrightarrow H$ . Moreover, if  $H$  is dodecahedrally connected,  $G_2$  can be chosen to be a handle expansion. By (7.4) and (7.5) there exist a handle or circuit expansion  $G'$  of  $G$  and a homeomorphic embedding  $\eta' : G' \hookrightarrow H$ . Moreover, if  $H$  is dodecahedrally connected, then  $G'$  is a handle expansion. Thus the pair  $G', \eta'$  satisfies the conclusion of the theorem.  $\square$

## ACKNOWLEDGEMENT

We thank Daniel P. Sanders for carefully reading the manuscript, and for providing helpful comments.

## REFERENCES

1. E. R. L. Aldred, D. A. Holton, B. Jackson, Uniform cyclic edge connectivity in cubic graphs, *Combinatorica* **11** (1991), 81–96.
2. D. Barnette, On generating planar graphs, *Discrete Math.* **7** (1974), 199–208.
3. J. W. Butler, A generation procedure for the simple 3-polytopes with cyclically 5-connected graphs, *Canad. J. Math.* **26** (1974), 686–708.
4. K. Edwards, D. P. Sanders, P. D. Seymour and R. Thomas, Three-edge-colouring doublecross cubic graphs, [arXiv:1411.4352](https://arxiv.org/abs/1411.4352).
5. W. McCuaig, Edge-reductions in cyclically  $k$ -connected cubic graphs, Ph. D. thesis, University of Waterloo, Waterloo, Ontario, October 1987.
6. W. McCuaig, Edge-reductions in cyclically  $k$ -connected cubic graphs, *J. Combin. Theory Ser. B* **56** (1992), 16–44.
7. N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, The four-colour theorem, *J. Combin. Theory Ser. B* **70** (1997), 2–44.
8. N. Robertson and P. D. Seymour, Graph Minors IX. Disjoint crossed paths, *J. Combin. Theory Ser. B* **49** (1990), 40–77.
9. N. Robertson, P. D. Seymour and R. Thomas, Tutte’s edge-coloring conjecture, *J. Combin. Theory Ser. B* **70** (1997), 166–183.
10. N. Robertson, P. D. Seymour and R. Thomas, Excluded minors in cubic graphs, [arXiv:1403.2118](https://arxiv.org/abs/1403.2118).

11. W. T. Tutte, Convex representations of graphs, *Proc. London Math. Soc.* **10** (1960), 304–320.
12. W. T. Tutte, A geometrical version of the Four Colour Problem, “Combinatorial Mathematics and its Applications”, Bose and Dowling Eds., *The University of North Carolina Press* (1969), 553-560.

This material is based upon work supported by the National Science Foundation. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.